Unsmoothing Real Estate Returns: A Regime-switching Approach

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Abstract

We propose newly developed unsmoothing techniques for appraisal-based real estate returns which are based on a regime-switching Threshold Autoregressive (TAR) model. We first examine analytically the conventional unsmoothing technique – which usually models the true returns by a linear Autoregressive (AR) process – and show that when true returns follow a TAR process, the conventional technique is misspecified, and hence underestimates the true variance. We argue that misspecification of the true returns result in the unsmoothed returns still being "too smooth". The approach also solves an identification problem suffered by the conventional method. Two exogenous variables, returns on the FT index and GDP growth, tend to outperform other variables as a regime indicator, with both delivering relatively low sum of squared errors (SSE). Furthermore, they appear to capture risks of downturns in real estate returns relatively well. We extend our regime-switching idea to the smoothing equation, thereby allowing for the switching behaviour by the appraiser. Doing so results in two new techniques, the TAR-AR and TAR-TAR approaches. The last 'co-switching' specification, in particular, opens up to a new frontier of empirical research. We estimated the TAR-TAR using the FT returns as the regime indicator, and found results that outperform conventional smoothing models and have plausible economic explanations

Keywords: Unsmoothing technique; Identification; Regime switching; Threshold Autoregressive model; Real estate returns.

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1. Introduction

Reported real estate returns differ from those of financial assets in that they rely on appraisals to measure periodic capital growth and income return. It has been argued that this practice results in "smoothing" of the reported returns. The temporal aggregation and lagging effects produce serial correlation in return series, and dampen reported volatility measures. In turn, this has implications for the use of real estate indices in asset allocation and performance measurement applications (Quan & Quigley 1991[1], Geltner *et al.* 2003).

Conventional appraisal-based unsmoothing methodology (Geltner 1991, 1993, Fisher *et al.* 1994, Cho *et al.* 2003, Booth & Marcato 2004, Marcato & Key 2007a,b) has been fairly successful in the sense that it generates higher volatility in the unsmoothed returns than in the observed 'smoothed' returns, and tends to reduce lagging effects when compared to public listed real estate indices. These results have influenced investor attitudes on the risk of real estate as an asset class. Further confirmation of this is provided by the development of repeat sales transaction-based indices such as the MIT series for the US, which show significantly higher variance of returns (see e.g. Fisher *et al.* 2003).

Nevertheless, a number of studies have shown that private real estate returns still appears to have significantly better risk-hedging characteristics than other asset classes (e.g. Hudson-Wilson *et al.* 2003, Worzala & Sirmans 2003, Bond *et al.* 2007b). Measures of volatility for unsmoothed series still seem too low in relation to the returns of financial assets and the use of unsmoothed data in conventional asset allocation optimisers produces weightings that out of line with professional investor practice. This has been attributed to an additional *ex ante* liquidity premium that is not accounted for in conventional returns (Bond *et al.,* 2007a), to investors' inability to diversify away specific risk fully due to large lot size (Baum, 2009) or to the distributional characteristics of real estate returns (Young, 2007).

In this paper, we argue that the conventional unsmoothing methodology is not completely satisfactory because it ignores non-linearity in the data and regime-switching behaviour in particular. Application of regime-switching models in financial markets is not uncommon (for example there are applications to stock returns (Li & Lam 1995), to exchange rate changes (Alba & Park 2005), and to property returns (Lizieri *et al.* 1998). Regime switching behaviour seems highly plausible for real estate returns which exhibit episodes of booms and busts due to the

[1] Quan and Quigley seek to identify an "optimal" appraisal in the face of transaction noise; subsequent work has tended to seek the "underlying" market return from the smoothed appraisal signal.

cyclical nature of property and credit markets. Instead of modelling the underlying real estate returns as an ARMA process, as in the previous studies, we employ a family of Threshold Autoregressive (TAR) models (Tong 1978, 1990), in effect, allowing for some non-stationarity. TAR models have been used in real estate applications previously (Lizieri *et al.* 1998, Brooks & Maitland-Smith 1999): we provide an extension and application to return measurement.

Our prior expectation is that the unsmoothing methodology based on TAR models will provide evidence of additional "built-in" volatility into real estate returns. In earlier research, Chaplin (1997) attempted to incorporate regimes into the unsmoothing methodology. He assumed that real estate returns were normally distributed, and divided them into six regimes with predetermined unsmoothing parameters (his theoretical framework follows Quan and Quigley's approach, but the values are asserted not estimated)[2]. Our methodology is more general in two main aspects. First, regimes may be defined in terms of property returns themselves (e.g. into periods of high, average, and low returns as Chaplin (1997) does) or in terms of exogenous variables driving property performance such as macroeconomic factors, credit conditions, and similar factors. Second, the threshold value can be estimated, rather than imposed.

The rest of the paper is organised as follows. Section 2 reviews the base model for the conventional unsmoothing technique. Then in Section 3 we set out how regimes can be incorporated into the unsmoothing technique via TAR models. In support of the TAR approach, we also present a misspecification analysis which shows that the conventional technique underestimates the true variance provided that the true returns follow a TAR process. Section 4 provides empirical results of the proposed technique. Section 5 examines implications on asset allocation exercises. Finally, we conclude and suggest further research.

[2] We should note also Brown & Matysiak (1997) who offer a smoothing model with a time varying alpha based on rolling window serial correlations.

2. Understanding the Base Model

2.1 The Measurement Equation

Consider a simple smoothing model:

$$r_t^* = \alpha r_{t-1}^* + (1-\alpha) r_t \,, \tag{1}$$

where r_t^* denotes the reported valuation-based return at t, r_t the "true" underlying return, and α the smoothing parameter – which is a weight given to information about the prior valuation, $\alpha \in (0,1)$. From Equation (1), given the value for the smoothing parameter, the unsmoothed returns can be computed by

$$r_t = \frac{1}{1-\alpha}\left(r_t^* - \alpha r_{t-1}^*\right). \tag{2}$$

It is crucial for us to thoroughly understand the measurement equation. It gives an implicit relationship between α and the unsmoothed volatility.

$$\operatorname{var}(r_t) = \frac{\operatorname{var}(r_t^*)\left[1-2\alpha\rho+\alpha^2\right]}{\left(1-\alpha\right)^2}, \tag{3}$$

where ρ is the first-order autocorrelation coefficient of r_t^* which is observable. It can be shown that

$$\left.\frac{\partial \operatorname{var}(r_t)}{\partial \alpha}\right|_{\alpha\in(0,1)} = \frac{2\operatorname{var}(r_t^*)\left(1-\rho\right)\left(1-\alpha^2\right)}{\left(1-\alpha\right)^4} > 0\,. \tag{4}$$

The result shows that the implied variance is increasing in α , when the parameter lies between zero and one. Artificially high values of alpha would inflate the variance, while too low values would understate volatility in the underlying return series. This may be important where researchers "assume" a value of alpha to desmooth an appraisal based series or where mis-estimation of alpha occurs. This idea can be easily extended to the generalised measurement equation, which includes more than a single lagged value of the observed returns.[3]

It is clear that, at this point, the variance implied by the smoothing equation has no direct relationship with the assumed "true" returns. When the smoothing coefficient is known, it does

[3] See Appendix A.

not matter what assumption is made regarding the true return generating process. The true returns generating process comes into the picture only when we attempt to estimate α.

2.2 The State Equation

The conventional method assumes that the true (unobservable) returns follow a stationary AR(1) process:

$$r_t = \theta + \phi r_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim iid(0, \sigma_\varepsilon^2), \tag{5}$$

where $|\phi| < 1$ by assumption.

Using Equations (1) and (5), and given the value for ϕ, the weight α can be estimated consistently by a recursive procedure from an implied model of the observed returns

$$r_t^* = \theta(1-\alpha) + (\alpha + \phi) r_{t-1}^* - \alpha\phi r_{t-2}^* + v_t, \tag{6}$$

where $v_t = (1-\alpha)\varepsilon_t$. This residual term assumes ϕ as given, and varies with α alone, i.e. $v_t = v_t(\alpha;\phi)$. The least squares method gives

$$\hat{\alpha} = \arg\min \sum_T v_t^2(\alpha;\phi). \tag{7}$$

The unsmoothed returns will then be given by

$$r_t = \frac{1}{1-\hat{\alpha}}\left(r_t^* - \hat{\alpha} r_{t-1}^*\right). \tag{8}$$

A new ϕ is a least squared estimate of Equation (5),

$$\hat{\phi} = \arg\min \sum_T \varepsilon_t^2(\phi;\alpha), \tag{9}$$

where at the moment the residual $\varepsilon_t = \varepsilon_t(\phi;\alpha)$ varies with ϕ only. The recursion continues until $\hat{\phi}$ converges – that is, when it differs from the previous value by an insignificant amount, e.g. 0.01 (Cho *et al.* 2003). The same estimation procedure still applies to extensions of the simple smoothing model, for example, to the inclusion of more lagged values of valuation based returns in the measurement equation, or to the generalization of the true returns to an ARMA process.

Now let us consider the variance of the reconstructed estimate of underlying returns. By construction,

$$\operatorname{var}(r_t) = \frac{\sigma_\varepsilon^2}{1-\hat{\phi}^2}. \tag{10}$$

This has been argued to be too small a value as to reflect actual risks in the underlying asset. As will be shown shortly, assuming the true returns follow a TAR process provides "built-in" volatility, while the usual iterative estimation procedure is still applicable.

3. TAR Models

Here we set out how a regime-switching approach can be incorporated into smoothing models. First, we will review the TAR model and its properties. Then we will show how to implement the regime-based unsmoothing methodology.

3.1 TAR Process and Properties

Suppose now that the true returns follow a threshold autoregressive process, where we suppress the intercepts to avoid excessive algebra:

$$r_t = \begin{cases} \phi_1 r_{t-1} + \varepsilon_t, & z_{t-1} > c, \\ \phi_2 r_{t-1} + \varepsilon_t, & z_{t-1} \le c, \end{cases} \tag{11}$$

The model can be written succinctly as

$$r_t = \left(\phi_1 I_{t-1} + \phi_2 (1 - I_{t-1})\right) r_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim iid(0, \sigma_\varepsilon^2). \tag{12}$$

Here z_t is an observable (weakly) exogenous regime indicator, with the corresponding indicator function $I_t = 1(z_t > c)$. We will call subsequently refer to ϕ_1 as the 'high state' coefficient, and to ϕ_2 as the 'low state' coefficient. This non-linear model, initially proposed by Tong (1978), splits the time series of interest into subsets, or 'regimes' defined with respect to the value of some regime indicator.[4] The variable z_t can be one of a range of variables known at time t.

Given c, the coefficients ϕ_1 and ϕ_2 can be estimated by applying OLS to Equation (12). Otherwise, the threshold value can be estimated empirically as

$$\hat{c} = \arg\min_{c \in C} \hat{\sigma}(c), \tag{13}$$

[4] Further discussion of TAR, as well as other regime-switching models, can be found in Tong (1990). Franses and van Dijk (2000) provide an excellent textbook treatment on the subject.

where C represents the set of all allowable threshold values, and $\sigma(c)$ the standard error of regressions given the threshold value. A popular choice of C that ensures consistency requires that each regime contains at least 15% of the number of observations (Franses and van Dijk, 2000). Other goodness of fit measures, e.g. the Akaike Information Criterion or Bayesian Information Criterion, may be used instead.

In the above specification, we can compute the variance of r_t using the Law of Iterated Expectation.

$$\mathrm{var}(r_t) = \frac{\sigma_\varepsilon^2}{1-\left(\pi\phi_1^2 + (1-\pi)\phi_2^2\right)}, \tag{14}$$

where π denotes the steady-state probability of the first regime.[5] Comparing this to the variance implied by the AR process in (10), we cannot tell whether or not the TAR approach will imply greater underlying volatility, as the relative magnitude of the two variances is not immediately clear. However, it is possible to show that, if the single-regime process is assumed, but the true returns do exhibit regime-switching behaviour, then the implied AR variance will be lower than the true variance. The misspecification analysis is presented in Appendix A, which demonstrates that, in large samples at least, the variance calculated by assuming an AR(1) process will be consistently underestimating the true variance.

3.2 Implementing the Unsmoothing Technique

Now we will present the implementation of our unsmoothing methodology, which is analogous to that of the conventional technique (see e.g. Cho *et al.* 2003). The main difference between the two techniques is that while the conventional one is linear, our model is non-linear.

Equations (1) and (12) imply the following process in the observed returns

$$\left(1-\left(\phi_1 I_{t-1} + \phi_2(1-I_{t-1})\right)L\right)(1-\alpha L) r_t^* = v_t, \tag{15}$$

where L denotes a lag operator defined by $Lx_t = x_{t-1}$, and again $v_t = (1-\alpha)\varepsilon_t$. The recursion is initialised by the values of $\left(\phi_1^0, \phi_2^0, c^0\right)$. We then estimate the smoothing α in Equation (15), given particular values of the three TAR parameters, by OLS. By using the estimated smoothing coefficient $\hat{\alpha}$ and the measurement equation (2), we can compute the unsmoothed returns.

[5] The derivation of this is shown in Appendix B.

These reconstructed estimates of the underlying returns will then be modelled as a TAR process. We estimate a new set of (ϕ_1, ϕ_2, c) as described in Section 3.2, with the estimated threshold value being that which minimises the standard error of regression.[6] The new set of $\left(\phi_1^1, \phi_2^1, c^1\right)$ will then be used in the next round of estimation, and the recursion stops when (ϕ_1, ϕ_2) converge in value. We will call this a AR-TAR process.

The formulation described above assumes that there exist different return regimes but a single smoothing process and, hence, a single value of alpha. However, there may also exist "smoothing regimes" where appraiser behaviour differs. For example, there may be periods characterized by thin trading (typically these will be periods when prices are falling, as owners with discretion may chose not to crystallize losses and retain their properties): in the absence of dense transaction evidence, appraisers may be more prone to smooth than in market environments with rich comparable evidence. While such smoothing regimes may coincide with the return process regimes, there is no reason why they *must* coincide. Accordingly, we separately define smoothing regimes via a TAR process, producing a double-TAR, or TAR-TAR process. Generalising, we define:

Smoothing Equation:
$$r_t^* = \alpha_t r_{t-1}^* + (1-\alpha_t) r_t, \tag{16}$$

Returns Process:
$$r_t = \gamma_t + \phi_t r_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim iid(0, \sigma_\varepsilon^2). \tag{17}$$

Here r_t^* denotes the observed "smoothed" returns, r_t the actual returns, ε_t the residual in the returns process, α_t the smoothing coefficient, γ_t the intercept term, and ϕ_t the persistence coefficient. We allow the parameters to be regime-switching – hence the time subscript – according to certain exogenous variables z_1 and z_2 as follows.

$$\alpha_t = \begin{cases} \alpha_1, & z_{1t-1} > c_1, \\ \alpha_2, & z_{1t-1} \le c_1. \end{cases} \tag{18}$$

$$(\gamma_t, \phi_t) = \begin{cases} (\gamma_1, \phi_1), & z_{2t-1} > c_2, \\ (\gamma_2, \phi_2), & z_{2t-1} \le c_2. \end{cases} \tag{19}$$

[6] In estimating the threshold value c, we use a grid search in our estimation since the search domain is of low dimensionality.

As before, the specification concerns the lagged value of the exogenous variables, and hence the current regimes are observable. Parameters are estimated iteratively using a grid search technique to identify the lowest sum of squared errors as in the AR-TAR formulation in (15)[7].

4. Data

Private real estate returns, defined as r_t represent the log difference of the IPD UK Total Return Index for all property. We utilize monthly data from December 1986 to December 2008, thus including the onset of the market correction at the end of the period. While the IPD monthly index does not completely track the IPD annual index, it represents institutional and professional investor holdings of real estate with a capital value of £32.5billion as at December 2008 and with in excess of 3,500 properties. Our analysis, however, utilizes quarterly returns. First, a number of the macro-economic regime indicators are only available quarterly; second, since monthly valuations frequently represent a simple desk-based update, and with greater information available on a quarterly basis, this may represent a more robust frequency compared to the more noisy monthly series.

The choice of exogenous regime indicators to be tested was based on prior research on the drivers of private real estate returns. We include an interest rate variable, three month LIBOR (end of month); most studies indicate that, as expected, real estate returns are strongly influenced by interest rates. Indeed, prior applications of TAR models in real estate (Lizieri *et al.,* 1998; Brooks and Maitland-Smith, 1999) both use real interest rates to determine thresholds. Lizieri *et al.* suggest that in the high interest rate environment, real estate exhibits greater volatility and sharply falling values.

Many models of real estate rents and capital values utilize an aggregate demand measure: as a result, we test UK GDP growth (available only as a quarterly series). This enables us to test whether property and/or appraiser behaviour differs in boom and recessionary periods. As a further indicator of macro-economic conditions, we examine service sector employment, which might proxy for space demand. We also use a financial market indicator, the FT All Share Total Return index. This can be justified as an extension of the market model (Ling and Naranjo 1997, 2000, Wike and Gillen 2008); furthermore, given the growing attention on tail dependence (and, in particular, asymmetric tail dependence – see Knight *et al.* 200x for a real estate example), upward and downward spikes in equity prices may be associated with capital market conditions that are adverse or positive for real estate.

We include a property market indicator, the initial yield (the ratio of rent payable to capital value). This is, in part, endogenous, in that the yield represents the cash return on investment and changes in yield (in effect the capitalization rate) drive shifts in capital values. However, with the growing attention on credit cycles, asset bubbles and the role of real assets as collateral, movements of the yield away from long-run average values might indicate that prices have moved above or below their fundamental economic values, presaging a correction. Given that part of the case made for property lies in its supposed "inflation hedging" properties (although evidence for this is mixed, particularly regarding unexpected inflation: see Hoesli *et al.* 2007 for a review), we include the retail price index as a measure of UK inflation. Finally, given that real estate investment is increasingly global and because changes in the exchange rate reflect expectations regarding national economic performance, we examine the USD-GBP exchange rate.

Descriptive statistics for these variable s are shown in Table 1 below.

Table 1: Descriptive Statistics

Variables	Mean	Median	Max	Min	S.D.	Skewness	Kurtosis
r_t	2.15	2.53	8.03	-14.51	3.25	-1.98	10.67
LIBOR	7.20	6.04	15.25	2.83	3.18	1.20	3.44
Inflation	0.80	0.75	2.73	-0.57	0.63	0.85	4.22
Real interest	6.39	5.55	13.81	2.67	2.88	1.10	3.29
Initial yield	6.74	6.96	9.09	4.57	1.18	-0.06	2.16
Exchange rate	1.68	1.65	2.04	1.41	0.16	0.45	2.32
FT returns	2.02	3.53	18.84	-32.00	8.59	-1.08	5.31
GDP	0.60	0.65	2.20	-1.80	0.57	-1.15	7.01
Employment	29,240	29,030	31,661	26,762	1,386	0.24	1.85

[7] We have also estimated models where the smoothing parameter varies by regime but where there is a single returns generating process – a TAR-AR model. These results are shown in appendix X: we do not discuss them here for reasons of length.

5. Estimation Results

5.1 AR and AR-TAR models.

First, we present the quarterly results of the TAR models and the base case AR model in Table 2. The estimated values for α and ϕ from the AR method are in line with the literature – the smoothing parameter is greater than half, while the lagged coefficient is relatively small in size, denoting small but non-zero autocorrelation in returns. For the TAR models, smoothing parameters vary between 0.1 (employment) and 0.9 (exchange rate) depending on the regime indicator. The smaller figure is the exception; most models show the expected high levels of smoothing. The best performing TAR models, as measured by size of sum-of-squares errors, are the FT returns, LIBOR and GDP models (employment has a relatively low SSE, but is not analysed further given the insignificant smoothing parameter). In terms of SSE, the best performing model is that defined by FT returns, with an SSE that is 42% lower than the base case AR model.

Figure 1 shows quarterly time-series plots, between 1986Q4 and 2008Q4, of log-returns on IPD total return index and the three best performing exogenous variables, LIBOR, log-returns on FT index, and quarterly GDP growth. The threshold value of each regime indicator is shown by the horizontal line. There have been two important crises in the UK real estate market, namely the 1990s crisis, and the recent financial crisis (2007-2009). A good regime indicator should thus be able to pick these up. Figure 1 below illustrates this point. While LIBOR managed to capture only the 1990s downturn, GDP growth captures the recent one too. However, GDP seems to respond more slowly than the equity index – it takes some quarters before GDP switches to the bad state. FT returns, on the other hand, seem to be a pretty good regime indicator. This variable does not only capture the two important downturns in the real estate market, but also other smaller downturns. FT returns also respond much faster than GDP does as the stock index is regarded as a leading indicator.

Table 2: Estimation Results for AR-TAR (Switching Return)

Model	α	γ_1	ϕ_1	γ_2	ϕ_2	c	π	[Min,Max]	SSE
TAR									
LIBOR	0.51** (0.07)	-1.25* (0.48)	1.27** (0.10)	2.38** (0.65)	0.18 (0.11)	6.25	0.56	[2.83,15.25]	233.45
Inflation rate	0.77** (0.09)	0.25 (1.67)	-0.09 (0.15)	-0.22 (1.08)	0.95** (0.23)	0.94	0.33	[-0.57,2.73]	259.44
Real interest	0.72** (0.14)	-0.98 (0.75)	0.98** (0.12)	3.01** (1.13)	-0.17 (0.09)	5.12	0.57	[2.67,13.81]	242.68
FT returns	0.53** (0.09)	2.22** (0.32)	0.31** (0.08)	-4.05** (0.94)	1.77** (0.24)	-1.54	0.76	[-32.00,18.84]	179.91
Exchange rate	0.93** (0.02)	-40.54 (235.98)	1.70 (7.65)	1.33 (2.20)	0.02 (0.10)	1.95	0.05	[1.41,2.04]	240.19
Initial Yield	0.92** (0.04)	1.60 (2.18)	0.02 (0.10)	-17.40** (3.85)	1.76** (0.32)	4.64	0.95	[4.57,9.09]	255.33
GDP Growth	0.81** (0.12)	1.72 (0.99)	0.11 (0.08)	5.18 (6.66)	4.27* (1.86)	-0.41	0.94	[-1.80,2.20]	208.52
Employment	0.08 (0.11)	-8.52 (8.05)	-0.32 (1.53)	0.41 (0.37)	0.83** (0.12)	31,414	0.06	[26684,31661]	208.20
AR	0.94** (0.04)	-1.35 (2.82)	0.12 (0.15)						309.53

Notes: i) the parameters $(\alpha,\phi_1,\phi_2,\gamma_1,\gamma_2,c,\pi)$ denote the smoothing coefficient, the 'high state' coefficient, the 'low state' coefficient, the 'high state' intercept, the 'low state' intercept, the threshold value, and the probability of high state; ii) [Min,Max] refers to the minimum and maximum values of the exogenous variables; (iii) Newey-West Heteroscedasticity and Autocorrelation consistent (HAC) standard deviations are reported in parenthesis; (iv) * denotes significance at 5%, ** at 1%. To ensure sufficient observations, the search is restricted between 5th and 95th percentiles.

In the LIBOR regime, the "high state" corresponds to interest rates in excess of 6.25%. In this state returns are negative and ϕ is strongly positive, implying typically falling values. In the lower interest rate regime, the intercept is significantly positive and the autoregressive term insignificantly different from zero. This has a ready economic interpretation, with returns adversely affected by high interest rates, with steady growth in more benign environments. The smoothing term is, however, relatively low compared to other models including the base case AR formulation. The GDP-based regime is somewhat harder to interpret: the only significant coefficient other than the smoothing parameter is the ϕ value for the low state, which is strongly positive and of large magnitude: since GDP is falling in the low state, this suggests sharp falls in value. However, the model is only in the low state 5% of the time. For the FT returns model, all coefficients are significant at the 0.01 level or beyond. When FT returns are above the threshold value, the intercept is positive and the ϕ term relatively small; in the low regime (when the equity market is falling), the intercept is strongly negative and ϕ is large, implying sharp falls. The world is in the low market state 24% of the time.

One of the interesting features of this analysis is that in many cases the autoregressive coefficient in one of the regimes is explosive, that is, it is bigger than one in absolute value. However, in these cases, the steady-state variance appears to exist, since the condition for the existence of an overall steady-steady variance can be satisfied even when there is no steady-state variance in one of the regimes; that is, $\phi_1^2\pi + (1-\pi)\phi_2^2 < 1$ and $\phi_1^2 > 1$ can occur simultaneously.[8] It is worth asking what is happening to returns when the process is in the high state; suppose that the process is in this regime for k consecutive periods, then, conditional on the time and value at entry, say period t, the variance at time $t+k$ will be

$$\text{var}(r_{t+k} \mid r_t, \text{ and being in the high state for } k \text{ consecutive periods}) = \sigma_\varepsilon^2 \sum_{j=0}^{k-1} \phi_1^{2j} . \qquad (16)$$

This can be seen to be exploding at the rate ϕ_1^{2k}. However, when we consider

$$\text{var}(r_{t+k} \mid r_t) = \pi^k \sigma_\varepsilon^2 \sum_{j=0}^{k-1} \phi_1^{2j} + \text{ extra terms}, \qquad (17)$$

we see that this explosive component is actually bounded by the condition that ensures the existence of an overall steady-state variance above. Figure 2 and Table 3 show fits of the AR and AR-TAR models and descriptive statistics for the FT-return based regime analyses.

[8] The condition is due to Knight and Satchell (2009), which is a special case of the general conditions developed by Quinn (1982), Nicholls and Quinn (1982), Feigin and Tweedie (1985), and Andel (1976).

Figure 1: Regime Indicators from AR-TAR Model

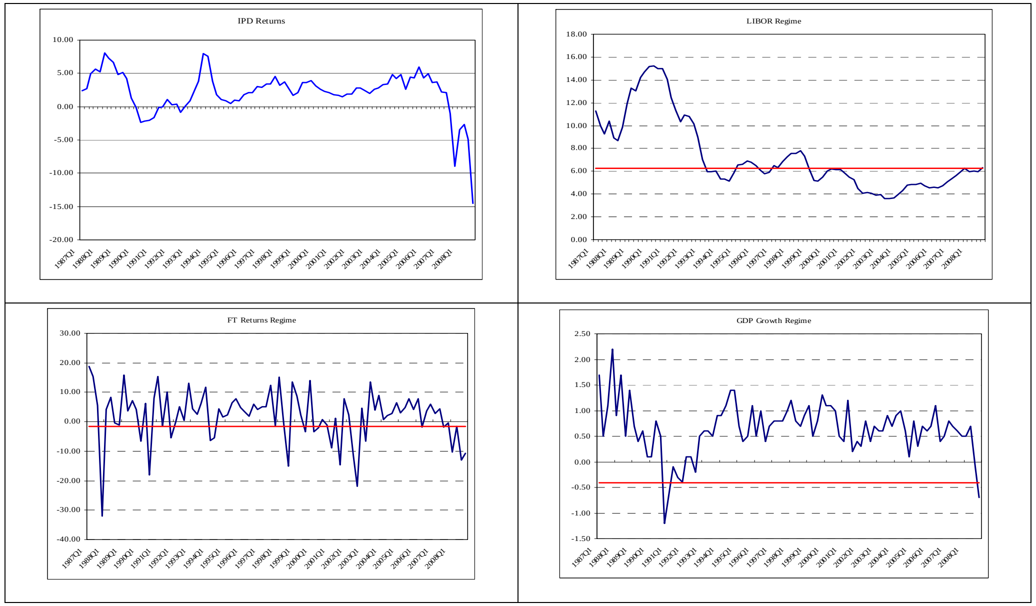


Figure 2: Fitted AR-TAR vs. AR

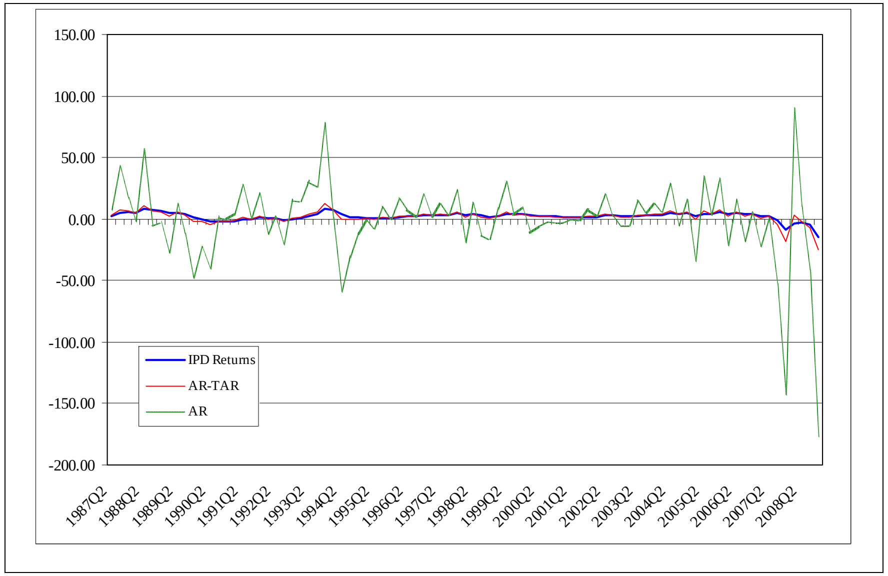


Table 3: Descriptive Statistics

	IPD Returns	AR-TAR	AR
Mean	2.15	1.92	-1.15
Median	2.53	2.22	1.86
Maximum	8.03	12.59	89.19
Minimum	-14.51	-25.36	-177.01
Std. Dev.	3.25	4.81	34.11
Skewness	-1.98	-2.79	-2.12
Kurtosis	10.67	16.44	13.16
Observations	88	87	87

5.2 The TAR-TAR Model

We restrict our analysis of the least restrictive version of our models, the TAR-TAR or co-switching model to the exogenous variables that proved most successful in the AR-TAR (and TAR-AR) models, FT returns and LIBOR. Table 4 presents results for four possible models: one with both regimes defined by interest rates, one with both regimes defined by equity returns and two models that mix FT returns and LIBOR as the determinants of the regimes. All four models show lower sum of squares errors than the base case AR model, with the model where both regimes are defined by FT returns (hereinafter FT-FT) exhibiting the lowest SSE, 41% below the base case result. The next best performing model has smoothing regime defined by equity returns but returns regime defined by LIBOR.

Examining, first, the FT-LIBOR model, smoothing appears to be more extreme in the low equity return regime – which only occurs when FT returns are falling very rapidly (the threshold value is -13%). For higher equity returns, smoothing, while still strongly significant, is below the level of the base case AR model and, hence, conventional estimates. The low FT high smoothing regime only occurs 8% of the time. The return regimes are determined by LIBOR; when interest rates are below the 6% threshold, ϕ is not significantly different from zero, with returns largely determined by the intercept, implying steady growth. In the higher interest rate environment, which one might expect to be associated with weaker real estate returns, the intercept is insignificant while the autoregressive term is significant at the 0.05 level. A combination of high smoothing (falling FT values) and high interest rates (significant auto-regression) suggests sharply falling returns in successive periods. However, this combination of regimes is rare, occurring just 2% of the time.

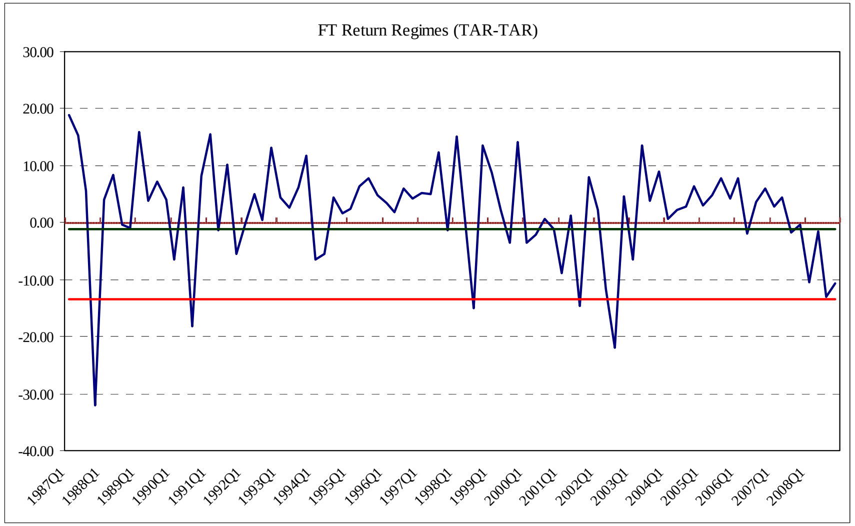
The FT-FT model identifies more extreme regimes, with both threshold values indicating falling equity values. The smoothing regime is defined by sharply falling FT prices: below the threshold, the smoothing parameter is higher, at 0.96, than above. The return process regime threshold is just negative at -1.2% (occurring 26% of the time). When equity returns are falling, the real estate intercept is strongly negative and there is significant and explosive auto-regression, suggesting sharply falling real estate returns. Above the threshold, the intercept is positive and ϕ insignificant, suggesting steady growth. Just over half the time, the market is in the steady growth, lower smoothing state; the stronger smoothing, falling return environment is rare, occurring just 7% of the time, identifying extreme states in the market.

Table 4: Estimation Results for TAR-TAR (Co-Switching)

Model	α_1	α_2	γ_1	ϕ_1	γ_2	ϕ_2	c_1	c_2	π_{ll}	π_{lh}	π_{hl}	π_{hh}	[Min,Max]	SSE
TAR														
LIBOR-LIBOR	1.42**	0.73**	-0.37	0.79**	3.05**	-0.04	6.21	11.31	0.55	0.00	0.33	0.12	[2.83,15.25]	250.27
	(0.28)	(0.07)	(1.33)	(0.14)	(0.68)	(0.15)								
FT-FT	0.72**	0.96**	3.36**	0.01	-7.13*	1.40**	-13.33	-1.20	0.07	0.00	0.19	0.74	[-32.00,18.84]	183.16
	(0.07)	(0.04)	(0.70)	(0.09)	(2.61)	(0.41)								
LIBOR-FT	1.40**	0.56**	1.63**	0.35**	5.28**	-0.85*	6.21	-1.79	0.15	0.40	0.09	0.36	as above	335.93
	(0.24)	(0.09)	(0.61)	(0.12)	(1.61)	(0.42)								
FT-LIBOR	0.79**	2.22**	0.24	0.59*	3.34**	-0.18	-13.34	5.95	0.02	0.06	0.41	0.51	as above	211.80
	(0.10)	(0.80)	(1.49)	(0.29)	(1.03)	(0.13)								
AR	0.94**		-1.35	0.12										309.53
	(0.04)		(2.82)	(0.15)										

Notes: i) the parameters $(\alpha_1, \alpha_2, \phi_1, \phi_2, \gamma_1, \gamma_2, c_1, c_2)$ denote the 'high state' smoothing coefficient, the 'low state' smoothing coefficient, the 'high state' coefficient, the 'low state' coefficient, the 'high state' intercept, the 'low state' intercept, the threshold value of the smoothing equation, and the threshold value of the returns process; ii) the probabilities $(\pi_{ll}, \pi_{lh}, \pi_{hl}, \pi_{hh})$ denote the state probability of both the smoothing equation and the returns process being in the low state, the probability of the smoothing equation being in the low state and the returns process in the high state, the probability of the smoothing equation being in the high state and the returns process in the low state, and the probability of both the smoothing equation and the returns process being in the high state; iii) [Min,Max] refers to the minimum and maximum values of the exogenous variables; (iv) the Newey-West Heteroscedasticity and Autocorrelation consistent (HAC) standard deviations are reported in parentheses; (v) * denotes significance at 5%, ** at 1%.

Figure 3: TAR-TAR in FT Returns



Note: the red line is the appraisal threshold, while the green line is the return threshold.

Figure 3 shows that once FT returns fall below -1.2%, then the return process shifts from the normal regime – where real estate returns have a positive mean and exhibit little persistence – to the bad or "crisis" regime – where returns have a negative mean and are highly explosive. However, it is not until FT returns fall below -13.33% that appraiser smoothing behaviour shifts regime. Figure 4 shows the fitted results. Panel A compares the index returns with the TAR-TAR and AR results. The extreme returns generated by the AR model obscure the relationship between the index returns and the TAR returns, which is shown more clearly in Panel B.

Figure 4: Fitted TAR Returns versus AR and Reported Data

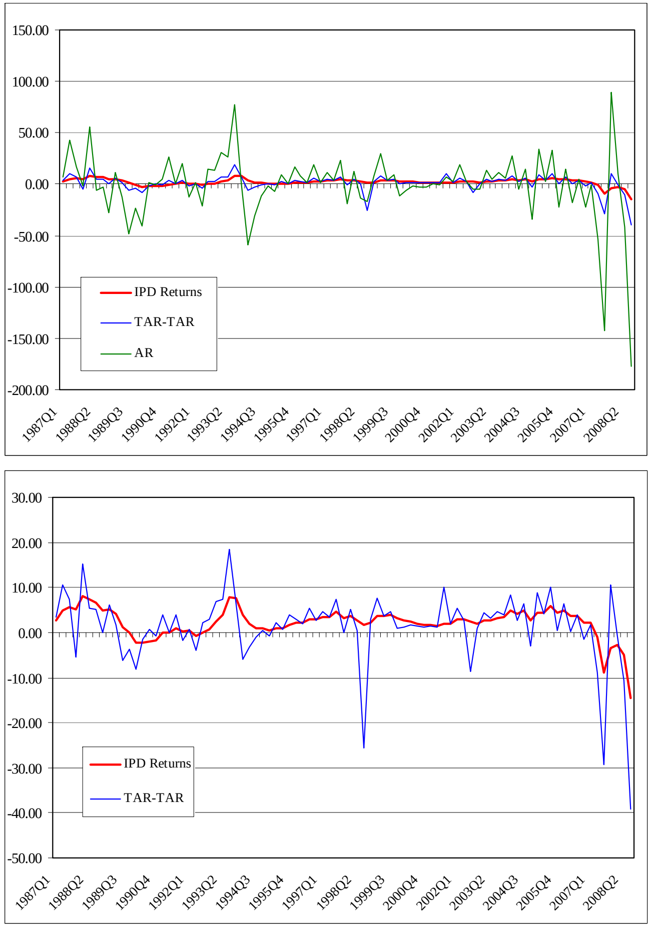


Table 5 shows descriptive statistics for the original appraisal-based index and the returns from the AR and TAR-TAR process. The AR process seems unsatisfactory, generating a negative mean return as a result of the extreme negative values at the end of the period (the median is positive) and with an infeasibly large standard deviation. The results from the TAR-TAR model seem intuitively more sound, with the standard deviation 2.4 times higher than the appraisal-based return and a reduction in the serial correlation. The final column of the table provides descriptive statistics for a “conventional” style desmoothing model as per equation 2, with alpha set equal to 0.8. The results are similar to the TAR-TAR model (the two series have a 0.92 correlation) but with a higher standard deviation. The conventional model suggests a fall of 60% in capital values in the second half of 2008, compared to a 45% for the TAR-TAR model and a reported fall of less than 20%.

Table 5 Descriptive Statistics ,Indexed, TAR-TAR and AR Models.

	IPD Returns	TAR-TAR	AR	α = 0.8
Mean	2.15	1.25	-1.15	1.36
Median	2.53	2.22	1.86	2.62
Maximum	8.03	12.59	89.19	24.42
Minimum	-14.51	-25.36	-177.01	-53.00
Std. Dev.	3.25	7.95	34.11	9.82
Skewness	-1.98	-2.79	-2.47	-2.72
Kurtosis	10.67	12.66	13.16	13.73
Serial Correlation	0.813	0.199	0.098	0.222
Observations	88	87	87	87

6. Conclusion

We have proposed a new unsmoothing technique for returns on an appraisal-based valuation index. The regime-switching TAR methodology not only allows us to distinguish the 'normal regime' variance from the unusual one, but also – and probably more importantly – fixes the identification problem encountered in the conventional AR method. For regimes, we utilise variables identified as significant in the determination of real estate returns. The most promising results come from the use of FT equity returns, interest rates (measured by LIBOR) and, to a lesser extent, by GDP. We examined models where the smoothing parameter was constant but the underlying return process varied by regime (AR-TAR), where the smoothing parameter changed but the returns process was time invariant (TAR-AR) and the least restrictive set of models where both smoothing parameter and returns process switched (TAR-TAR). The best models outperformed the base case single smoothing parameter AR process.

Of the TAR-TAR models, the best performing had both smoothing and returns process regimes determined by FT returns. When equity markets were falling, underlying real estate returns appear to behave differently than when they are rising: the intercept term is strongly negative and the autoregressive parameter exceeds one, implying sharply falling prices. Further, when equity prices are falling particularly sharply, smoothing increases: given that the model suggests that real estate returns are likely to be negative in this market environment, this may well be an information effect as transaction volumes fall[9]. This high smoothing, explosive regime is, perhaps fortunately, short-lived. It does point, though, to possible tail dependence between real estate and equity returns distributions, providing an interesting link to the emerging literature on tail dependence in asset markets. This has important implications for mixed-asset portfolio diversification strategies.

[9] There are a number of possible interpretations of this finding. One is that valuers "over-react" to falling prices, marking next period prices down further than is required; another is that observed transaction prices understate the true fall in values since owners do not bring properties to market, reducing transaction activity and masking falls.

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Appendices

Appendix A: Variance Manipulation in a Generalised Smoothing Equation

The idea of 'variance manipulation' is applicable to the extended smoothing equation, which depends on further lagged values of the observed returns:

$$r_t^* = \sum_{j=1}^{n} \alpha_j r_{t-j}^* + \alpha_{n+1} r_t , \tag{18}$$

where $\sum_{j=1}^{n} \alpha_j + \alpha_{n+1} = 1$. As a result, the unsmoothed returns and their volatility will be given respectively by

$$r_t = \frac{1}{1-\sum_{j=1}^{n} \alpha_j} \left(r_t^* - \sum_{j=1}^{n} \alpha_j r_{t-j}^* \right), \tag{19}$$

and

$$\operatorname{var}(r_t) = \left(\varpi' i\right)^{-2} \varpi' \Omega \varpi , \tag{20}$$

where an $(n+1)$ vector $\varpi = \left(1, -\alpha_1, ..., -\alpha_n\right)'$, an $(n+1)$ vector $i = \left(1,1,...,1\right)'$ and Ω an $(n+1)\times(n+1)$ autocovariance matrix of r^* whose element-ij is $\operatorname{cov}(r_{t-i+1}^*, r_{t-j+1}^*)$. Given knowledge of Ω, the weights ϖ can be calibrated such that the desirable level of $\operatorname{var}(r_t)$ is achieved.

Appendix B. Variance of TAR processes

We consider the following TAR model – written by using an indicator I_t –

$$y_{t+1} = \alpha_1 I_t + \alpha_2 (1-I_t) + \left[\beta_1 I_t + \beta_2 (1-I_t)\right] y_t + \left[\sigma_1 I_t + \sigma_2 (1-I_t)\right] v_{t+1} \tag{21}$$

where we seek a representation for the variance in terms of more fundamental parameters. It is assumed that the process $v_{t+1} \sim iid(0,1)$.

The regime indicator depends upon some variable z_t and we assume that

$$I_t = \begin{cases} 1 & \text{if } z_t > c, \\ 0 & \text{if } z_t \le c. \end{cases} \tag{22}$$

Without loss of generality, the threshold value c is assumed to be zero. The variable z_t can be a range of variables, known at time t; i.e., $z_t \in \Omega_t$. It could be independent of the process $\{v_t\}$, or it could be the process $\{y_t\}$. The two cases that we consider are discussed next. We assume that the process $\left(y_t, z_t\right)'$ is (jointly) weakly stationary, and that the first two moments exist.

To cover all such possibilities, we define the following parameters: $C^i = \operatorname{cov}(y_t^i, I_t)$; $\pi = prob(I_t = 1)$. The mean and variance will be denoted by μ and σ^2 respectively.

Taking unconditional expectations of **Error! Reference source not found.**, we see that

$$\mu = \alpha_2 + \left(\alpha_1 - \alpha_2\right)\pi + \beta_2 \mu + \left(\beta_1 - \beta_2\right)\left(C^1 + \mu\pi\right). \tag{23}$$

Rearranging **Error! Reference source not found.**, we see that

$$\mu = \frac{\alpha_2 + \left(\alpha_1 - \alpha_2\right)\pi + \left(\beta_1 - \beta_2\right)C^1}{1 - \left(\pi\beta_1 + (1-\pi)\beta_2\right)}. \tag{24}$$

This expresses the mean in terms of the state probabilities, the parameters of the dynamic process, and the covariance between the state and the regime variables. Equation **Error! Reference source not found.** shows that a TAR process could generate a non-trivial mean due to the presence of C^1, even if $\alpha_1 = \alpha_2 = 0$, a contrast to a simple AR(1) without drift model. In the special but important case of $C^1 = 0$, which results from the standard exogeneity assumption, we see that

$$\mu = \frac{\pi\alpha_1 + (1-\pi)\alpha_2}{1-(\pi\beta_1 + (1-\pi)\beta_2)}. \tag{25}$$

Equation **Error! Reference source not found.** is very satisfactory, since it depends only on the steady-state probability of z (a marginal concept) and the fundamental parameters of **Error! Reference source not found.**. Equation **Error! Reference source not found.** is less so as the covariance term could be constructed as containing information about the y distribution, that is, it could be thought of as endogenous rather than exogenous.

We shall now repeat our analysis for the variance of y. By the Law of Iterated Expectation,

$$\sigma^2 = E[\mathrm{var}(y_{t+1} \mid w_t)] + \mathrm{var}(E[y_{t+1} \mid w_t]) \tag{26}$$

where $w_t = (y_t, I_t)'$. The expression in **Error! Reference source not found.** is simply a result of the Law of Iterated Expectation.

$$\mathrm{var}(y_{t+1} \mid w_t) = (\sigma_1 I_t + \sigma_2(1-I_t))^2. \tag{27}$$

Hence,

$$E[\mathrm{var}(y_{t+1} \mid w_t)] = \pi\sigma_1^2 + (1-\pi)\sigma_2^2. \tag{28}$$

$$E[y_{t+1} \mid w_t] = \alpha_2 + (\alpha_1 - \alpha_2)I_t + \beta_2 y_t + (\beta_1 - \beta_2)I_t y_t. \tag{29}$$

Hence,

$$\begin{aligned} \mathrm{var}(E[y_{t+1} \mid w_t]) = {} & (\alpha_1-\alpha_2)^2\pi(1-\pi) + \beta_2^2\sigma^2 + (\beta_1-\beta_2)^2\,\mathrm{var}(I_t y_t) \\ & + 2(\alpha_1-\alpha_2)\beta_2\,\mathrm{cov}(I_t, y_t) + 2(\alpha_1-\alpha_2)(\beta_1-\beta_2)\,\mathrm{cov}(I_t, I_t y_t) \\ & + 2\beta_2(\beta_1-\beta_2)\,\mathrm{cov}(y_t, I_t y_t). \end{aligned} \tag{30}$$

The covariances can be written in terms of the fundamental parameters as

$$\mathrm{cov}(I_t, y_t) = C^1;$$

$$\mathrm{cov}(I_t, I_t y_t) = (C^1 + \pi\mu)(1-\pi);$$

$\mathrm{cov}(y_t, I_t y_t) = C^2 + (\sigma^2 + \mu^2)\pi - (C^1 + \pi\mu)\mu$; and $\mathrm{var}(I_t y_t) = C^2 + (\sigma^2 + \mu^2)\pi - (C^1 + \pi\mu)^2$.

Substituting **Error! Reference source not found.** and **Error! Reference source not found.** into **Error! Reference source not found.**, we get

$$\begin{aligned} \sigma^2 = \frac{1}{1-(\pi\beta_1^2 + (1-\pi)\beta_2^2)} \{ & \pi\sigma_1^2 + (1-\pi)\sigma_2^2 + (\alpha_1-\alpha_2)^2\pi(1-\pi) \\ & + 2[(\alpha_1-\alpha_2)(\beta_1-\beta_2)\pi(1-\pi)]\mu \\ & + [(\beta_1-\beta_2)^2\pi(1-\pi)]\mu^2 \\ & - 2[(\beta_1-\beta_2)(\pi\beta_1 + (1-\pi)\beta_2)]\mu C^1 \\ & + 2[(\alpha_1-\alpha_2)((1-\pi)\beta_1 + \pi\beta_2)]C^1 \\ & - (\beta_1-\beta_2)^2(C^1)^2 + (\beta_1^2 - \beta_2^2)C^2 \}. \end{aligned} \tag{31}$$

Together with **Error! Reference source not found.**, equation **Error! Reference source not found.** gives the variance of the variable y in terms of the fundamental parameters. Unlike the mean, the variance also depends on the "second moment" C^2; assuming uncorrelatedness does not ensure that this term vanishes.

Consider the case of exogeneity (or the stronger case of independence) which implies $C^1 = C^2 = 0$.

$$\begin{aligned} \sigma^2 &= \frac{1}{1-(\pi\beta_1^2 + (1-\pi)\beta_2^2)} \{ \pi\sigma_1^2 + (1-\pi)\sigma_2^2 + (\alpha_1-\alpha_2)^2\pi(1-\pi) \\ &\qquad + 2[(\alpha_1-\alpha_2)(\beta_1-\beta_2)\pi(1-\pi)]\mu \\ &\qquad + [(\beta_1-\beta_2)^2\pi(1-\pi)]\mu^2 \} \\ &= \frac{1}{1-(\pi\beta_1^2 + (1-\pi)\beta_2^2)} \{ \pi\sigma_1^2 + (1-\pi)\sigma_2^2 + \\ &\qquad + \pi(1-\pi)[(\alpha_1-\alpha_2) + (\beta_1-\beta_2)\mu]^2 \}, \end{aligned} \tag{32}$$

where μ is now given by **Error! Reference source not found.**. Equation **Error! Reference source not found.** shows that the unconditional variance consists of two terms: (i) a weighted average of the variance in each regime; and (ii) the unambiguously non-negative constant. When $\pi = 1$ (or 0), σ^2 becomes the single-regime AR variance $\frac{\sigma_1^2}{1-\beta_1^2}$ (or $\frac{\sigma_2^2}{1-\beta_2^2}$).

Appendix C: TAR-AR (Switching Smoothing) Results

Model	α_1	α_2	γ	ϕ	c	π	[Min,Max]	SSE
TAR								
LIBOR	1.22**	0.75**	3.69	-0.04	12.30	0.10	[2.83,15.25]	272.24
	(0.10)	(0.20)	(2.71)	(0.20)				
Inflation rate	0.79**	1.33**	2.21	-0.09	0.87	0.43	[-0.57,2.73]	269.40
	(0.14)	(0.11)	(1.97)	(0.18)				
Real interest	1.21**	0.75**	4.07	-0.05	5.35	0.53	[2.67,13.81]	272.31
	(0.09)	(0.21)	(2.97)	(0.21)				
FT returns	0.21**	1.97**	1.17	0.60**	-1.54	0.76	[-32.00,18.84]	195.51
	(0.08)	(0.25)	(0.67)	(0.16)				
Exchange rate	2.62	-0.03	-0.24	1.02**	1.96	0.05	[1.41,2.04]	255.38
	(1.41)	(0.14)	(1.54)	(0.34)				
Initial Yield	Not converged							
GDP Growth	0.82**	1.86**	1.78	0.08	-0.59	0.95	[-1.80,2.20]	246.65
	(0.12)	(0.45)	(2.90)	(0.12)				
Employment	Not converged							
AR	0.94**		-1.35	0.12				309.53
	(0.04)		(2.82)	(0.15)				

Notes: i) the parameters $(\alpha_1, \alpha_2, \phi, \gamma, c, \pi)$ denote the 'high state' smoothing coefficient, the 'low state' smoothing coefficient, the coefficient for lagged unsmoothed returns, the intercept term in the returns equation, the threshold value, and the probability of high state; ii) [Min,Max] refers to the minimum and maximum values of the exogenous variables; (iii) Newey-West Heteroscedasticity and Autocorrelation consistent (HAC) standard deviations are reported in parenthesis; (iv) * denotes significance at 5%, ** at 1%. To ensure sufficient observations, the search is restricted between 5th and 95th percentiles.

Appendix D: Identifiability

A further look at the results reveals, moreover, that the supposedly sensible parameters may be obtained by permuting (α, ϕ_1, ϕ_2). For quarterly LIBOR, for instance, a permuted set of parameters (0.94, 0.49, 0.10) seems to be more in line with the conventional belief than the original estimates (0.10, 0.49, 0.94). The former suggests smoothing, with less persistent "true" returns, while the latter suggests the opposite. Whilst the two lagged coefficients and the smoothing coefficient appear to be present somewhere among the three TAR-parameters, they are not where we would expect them to be conventionally. Therefore, we need to reexamine how we identify these parameters in the base model, and led us to question the identification of the base model itself.

Once estimation of the relevant parameters is concerned, identification becomes a critical matter. However, this is unfortunately left unexplored in the literature.

D.1 The Identification Problem in the Base Model

Identification of the base model parameters has been taken as granted, and the conventional recursive estimation procedure assumed valid. However, careful consideration actually reveals that the parameters $(\alpha, \phi, \sigma_\varepsilon^2)$ are not separately identifiable in the sense that we can infer specific values from our least-squares problem, i.e. in the sense of knowing which number is α and which is ϕ in the AR(1) smoothing model described by Equations (1) and (5).

We will illustrate this by looking at the likelihood function implied by the implied AR(2) process as in Equation (6). If the residuals in the AR(1) process are normally distributed, i.e. $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$, assuming that the first two observations are fixed and known, then the likelihood will be given by

$$\log L(r^*; \alpha, \phi, \sigma_\varepsilon^2) = -\frac{T-2}{2} \log 2\pi (1-\alpha)^2 \sigma_\varepsilon^2 - \frac{1}{2(1-\alpha)^2 \sigma_\varepsilon^2} \sum_{t=3}^{T} \left(r_t^* - (\alpha+\phi) r_{t-1}^* + \alpha\phi r_{t-2}^* \right)^2, \quad (33)$$

where the parameters in the arguments of the functions denote respectively the smoothing coefficient, the AR(1) coefficient, and the residual variance.

Now let us consider another model structure

$$r_t^* = \phi r_{t-1}^* + (1-\phi) r_t, \quad (34)$$

$$r_t = \alpha r_{t-1} + u_t, \quad u_t \sim iid(0, \sigma_u^2). \quad (35)$$

where the variance of the new residual u_t is assumed to be as in the equation below.

$$\sigma_u^2 = \left(\frac{1-\alpha}{1-\phi}\right)^2 \sigma_\varepsilon^2 . \tag{36}$$

Under normality, the likelihood of this structure will be given by

$$\log L(r^*;\phi,\alpha,\sigma_u^2) = -\frac{T-2}{2}\log 2\pi\left(1-\phi\right)^2\sigma_u^2 - \frac{1}{2\left(1-\phi\right)^2\sigma_u^2}\sum_{t=3}^{T}\left(r_t^* - \left(\alpha+\phi\right)r_{t-1}^* + \alpha\phi r_{t-2}^*\right)^2 . \tag{37}$$

By substituting for the value for σ_u^2, we found that

$$\log L(r^*;\alpha,\phi,\sigma_\varepsilon^2) = \log L(r^*;\phi,\alpha,\left(\frac{1-\alpha}{1-\phi}\right)^2\sigma_\varepsilon^2) . \tag{38}$$

Therefore, this new structure and the original one are observationally equivalent.

In practice, we usually suppose a high level of smoothing (e.g. from Table 1 $\alpha = 0.92$), but a low level of returns persistence ($\phi = 0.13$). Such supposition is not totally correct unless we have some additional prior information, for example, requiring that the smoothing coefficient should be greater than a half.[10] The identification analysis above shows that this might as well result from the contrary – the smoothing level is low, but the persistence level is high. We illustrate this point by calculating the sum squared errors (SSE) of the implied AR(2) equations using the permutated parameters from Table 2; the results are reported in Table D1. We see both the original set of parameters estimated by the recursive least squares $\left(\alpha,\phi\right) =$ (0.92, 0.13) and the permuted set (0.13, 0.92) give the same SSE of 309.83. This implies that both could possibly be an equally good representation of the smoothing model.

On the other hand, our TAR technique does not seem to suffer from an identification problem. For example, using LIBOR as a regime indicator, we obtain an SSE of 305.15 for the estimated set $\left(\alpha,\phi_1,\phi_2\right) =$ (0.10, 0.49, 0.94), 307.10 for (0.94, 0.49, 0.10) and 342 for (0.94, 0.10, 0.49); the last two combinations swap the smoothing and the autoregressive coefficients, and seem to be more in line with the conventional wisdom, while the first one gives the best fit to the data.[11] This finding supports the claim that the conventional belief could be faulty.

[10] Another example is to require the smoothing coefficient to be strictly positive, i.e. ruling out the case of anti-unsmoothing. Then the AR parameters are separately identifiable if one takes on a negative value.

[11] It is also clear from this LIBOR exercise that swapping the two autoregressive coefficients of the two regimes only increases SSE. We thus omit the results for the other regime indicators.

Nevertheless, the TAR method still seems to suffer from an estimation problem, due to its non-linearity. Clear, the estimated coefficients obtained from the recursive procedure do not always achieve the minimum SSE.[12] For example, for TAR with FT returns, the estimated parameters (0.25, 0.83, 1.69) gives higher SSE than (0.83, 0.25, 1.69) which actually seems more consistent with the conventional belief. In conclusion, although the implementation needs to be perfected, TAR does help with identification, and we elaborate this point in the next section.

Table D1: Identification Analysis

Quarterly	α	ϕ_1	ϕ_2	SSE
AR	0.92	0.13		309.83*
	0.13	0.92		309.83*
TAR				
LIBOR	0.10	0.49	0.94	305.15*
	0.10	0.94	0.49	495
	0.49	0.10	0.94	337
	0.49	0.94	0.10	492
	0.94	0.49	0.10	307.10
	0.94	0.10	0.49	342
Inflation rate	0.13	-0.28	0.93	309.78*
	-0.28	0.13	0.93	345
	0.93	-0.28	0.13	309.97
Real interest	0.09	0.93	1.26	309.09*
	0.93	0.09	1.26	310.31
	1.26	0.93	0.09	582.96
FT RET	0.25	0.83	1.69	278.75
	0.83	0.25	1.69	258.52*
	1.69	0.83	0.25	788
Exchange rate	0.01	8.58	0.94	249.50*
	8.58	0.01	0.94	15,381
	0.94	8.58	0.01	557
Initial Yield	0.09	0.96	0.78	306.42
	0.96	0.09	0.78	292.30*

[12] We could reasonably expect the sum squared function to possess many relative minima, and hence the recursive least squared method might not be able to select the global minimum, even though it still achieves convergence.

	0.78	0.96	0.09	423.37
GDP Growth	0.12	0.88	2.75	323.09
	0.88	0.12	2.75	311.91*
	2.75	0.12	0.88	3,242
Employment	0.92	0.05	0.15	309.61*
	0.05	0.92	0.15	928.83
	0.15	0.05	0.92	327.08

Notes: i) the first sets of parameters are those obtained from the recursive estimation; ii) the others are combinatorial; iii) the sum of squared errors (SSE) are computed from the residuals v_t of regressions (6) for AR method, and (15) for TAR; (iv) the minima are marked by *.

D.2 Identifiability in Unsmoothing with TAR

The exogenous regime indicator helps identify the autoregressive coefficients of the two regimes, and hence the smoothing coefficient. By the same line of analysis, under normality the likelihood function for Equation (15) is given by

$$\log L(r^*;\alpha,\phi_1,\phi_2,\sigma_\varepsilon^2) = -\frac{T-2}{2}\log 2\pi(1-\alpha)^2\sigma_\varepsilon^2 \\ -\frac{1}{2(1-\alpha)^2\sigma_\varepsilon^2}\sum_{t=3}^{T}\left(r_t^* - \left(\alpha + \left(\phi_1 I_{t-1} + \phi_2(1-I_{t-1})\right)\right)r_{t-1}^* + \alpha\left(\phi_1 I_{t-1} + \phi_2(1-I_{t-1})\right)r_{t-2}^*\right)^2. \tag{39}$$

The additional exogenous information pins down the regime-wise autoregressive parameters, i.e. swapping (ϕ_1,ϕ_2) will result in a different likelihood. A trivial condition that ensures identifiability is that the steady-state probability $\pi > 0$. It is also clear from the above expression that swapping the smoothing coefficient α with one of the autoregressive coefficient will change the likelihood, as the second term will change, in spite of the residual variance being scaled as shown in the previous section. This explains why the SSEs of the TAR models in Table 3 always vary with the combinations.

D.3 Alternative Solutions to the Identification Problem

We have seen that allowing the true returns to follow a different process help with identification. Nevertheless, the simplest way to facilitate identifiability is probably to impose some restrictions on the values of some of the parameters, e.g. the residual variance. For example, in the AR technique, one could simply set $\sigma_\varepsilon^2 = 1$, then the other two parameters (α,ϕ) will become separately identifiable.

An alternative, and plausibly better, approach to this is to re-specify the measurement equation. We add a noise term $\eta_t \sim iid(0,1)$ to the right hand side of Equation (1); this represents measurement errors in the valuation. The specification is similar in spirit to Geltner (1991), although in the previous work the author assumes that the noise term is diversified away at the aggregate-level data. The new measurement and the state equations imply a regression

$$(1-\alpha L)(1-\phi L)r_t^* = v_t + (1-\phi L)\eta_t, \tag{23}$$

where $v_t = (1-\alpha)\varepsilon_t$. This is an ARMA(2,1) process in r^*.

The MA part can be equivalently described by an MA(1) process

$$v_t + (1-\phi L)\eta_t \sim \xi_t + \delta\xi_{t-1}. \tag{24}$$

The new parameters match the primitive parameters according to the following relationships.

$$(1+\delta^2)\sigma_\xi^2 = (1-\alpha)^2\sigma_\varepsilon^2 + (1-\phi^2), \tag{25}$$

$$\delta\sigma_\xi^2 = -\phi. \tag{26}$$

The parameter δ will be determined by the triplet $(\alpha,\phi,\sigma_\varepsilon^2)$, and is never equal to either α or ϕ. That rules out the possibility of common factors, and hence the equation is identifiable. Moreover, this specification is superior to the first suggestion in a sense that it still allows us to identify the value for σ_ε^2 which is of economic significance.

D.4 Practical Solutions to the Identification Problem

Instead of assuming that the returns are demeaned – thus suppressing the intercept term that should have been in (5) – we could reintroduce the intercept term into the true returns process.

$$r_t = \gamma + \phi r_{t-1} + \varepsilon_t, \tag{40}$$

where γ is the intercept which has been so far assumed nil. As it turned out, doing so solves the identification problem, and improves the goodness of fit (though only marginally) altogether. The new estimation results are shown in Table 4, and the identification analysis in Table D2.

In Table B2, we see that our TAR approach suffered in estimation. The LS estimates could not always reach to the absolute maximum, and that the permuted set of estimates yielded lower SSE. This might be a symptom of model misspecification in which we assumed that the intercept values do not differ across regimes, and are equal to zero. We therefore include the intercept terms into the TAR, which results in

$$r_t = \begin{cases} \gamma_1 + \phi_1 r_{t-1} + \varepsilon_t, & z_{t-1} > c, \\ \gamma_2 + \phi_2 r_{t-1} + \varepsilon_t, & z_{t-1} \le c. \end{cases} \tag{41}$$

The results in Tables 4 and 5 confirm that this is a better model.

Table D2: Estimation Results for Models with Intercepts

Model	α	γ_1	ϕ_1	γ_2	ϕ_2	c	π	[Min,Max]	SSE
TAR									
LIBOR	0.11	-0.20	0.94	-0.42	1.31	4.57	0.85	[2.83,15.25]	302.35
	0.23	-0.93	1.21	1.05	0.64	6.06	0.51	[2.83,15.25]	257.50
Inflation rate	0.12	-0.14	0.74	-0.10	0.95	2.66	0.02	[-0.57,2.73]	309.33
Real interest	0.12	-0.11	0.94	1.38	0.70	3.00	0.98	[2.67,13.81]	308.23
FT returns	0.55	2.45	0.29	-4.22	1.64	-0.48	0.70	[-32.00,18.84]	183.63
Exchange rate	0.93	-40.54	1.70	1.33	0.02	2.00	0.02	[1.41,2.04]	240.19
Initial Yield	-0.01	0.54	0.81	-1.01	1.07	6.06	0.66	[4.57,9.09]	285.60
GDP Growth	0.11	0.38	0.81	7.45	4.49	-0.40	0.97	[-1.80,2.20]	208.52
Employment	0.25	-6.46	0.10	0.64	0.76	31,461	0.07	[26684,31661]	220.37
AR	0.94	-1.35	0.19						309.53

Notes: i) the parameters $(\alpha, \phi_1, \phi_2, \gamma_1, \gamma_2, c, \pi)$ denote the smoothing coefficient, the 'high state' coefficient, the 'low state' coefficient, the 'high state' intercept, the 'low state' intercept, the threshold value, and the probability of high state; ii) [Min,Max] refers to the minimum and maximum values of the exogenous variables.

Apart from the LIBOR case whose parameter values do not converge, the results in Table D2 are more reliable that those in Table 2 in terms of goodness of fit. Nevertheless, the estimates do not change vastly in value. The TAR on FT returns achieves the best fit. The identification analysis in Table B3 shows no problem of "relative minimum" which were previously achieved by the estimated coefficients.

The introduction of the intercepts also helps with the economic interpretation as well. For instance, consider the TAR on FT. Now it is clear that in the 'high state' (FT returns > -0.48%), the unsmoothed real estate returns exhibit a positive mean ($\gamma_1 > 0$), and show small persistence ($\phi_1 = 0.29$); on the other hand, in the 'low state', the unsmoothed real estate returns are explosive, with a negative mean returns. The estimated smoothing coefficient is now 0.55 with is noticeably larger than what we had before.

Table D3: Identification Analysis for Models with Intercepts

Quarterly	α	γ_1	ϕ_1	γ_2	ϕ_2	SSE
AR	0.94	-1.35	0.19			309.53*
	0.19	-1.35	0.94			417.21
TAR						
LIBOR	0.23	-0.93	1.21	1.05	0.64	257.50*
	0.64	-0.93	1.21	1.05	0.23	258.03
Inflation rate	0.12	-0.14	0.74	-0.10	0.95	309.33*
	0.74	-0.14	0.12	-0.10	0.95	390.42
	0.95	-0.14	0.74	-0.10	0.12	314.32
Real interest	0.12	-0.11	0.94	1.38	0.70	308.23*
	0.94	-0.11	0.12	1.38	0.70	309.70
	0.70	-0.11	0.94	1.38	0.12	381/80
FT RET	0.55	2.45	0.29	-4.22	1.64	183.63*
	0.29	2.45	0.55	-4.22	1.64	240.80

Notes: i) the first sets of parameters are those obtained from the recursive estimation; ii) the others are combinatorial; (iii) the minima are marked by *.

Appendix E Misspecification Analysis

Suppose that the true DGP is TAR(1) as in Equation (11). We retain all the assumptions, including the exogeneity of z_t. Now consider an OLS estimator of ϕ, ignoring the existence of regimes.

$$\hat{\phi} = \frac{\sum r_t r_{t-1}}{\sum r_{t-1}^2}. \tag{42}$$

By the Weak Law of Large Numbers of stationary mixing sequences (see McCabe and Tremayne, 1993), it follows that

$$\hat{\phi} \xrightarrow{p} \frac{E[r_t r_{t-1}]}{E[r_{t-1}^2]} = \pi\phi_1 + (1-\pi)\phi_2, \tag{43}$$

since $E[r_t r_{t-1}] = \pi\phi_1 E[r_{t-1}^2] + (1-\pi)\phi_2 E[r_{t-1}^2]$. Equation (43) is very intuitive – when the regime indicator is exogenous – the single regime coefficient is simply a weighted average of the coefficients for the two regimes. It follows that

$$Est.\operatorname{var}(r_t) \xrightarrow{p} \frac{\sigma_\varepsilon^2}{1-\left(\pi\phi_1 + (1-\pi)\phi_2\right)^2}. \tag{44}$$

The misspecified implied variance cannot be larger than the variance of the TAR process in (14) since $\pi\phi_1^2 + (1-\pi)\phi_2^2 \geq \left(\pi\phi_1 + (1-\pi)\phi_2\right)^2$. This is because

$$\pi\phi_1^2 + (1-\pi)\phi_2^2 - \left(\pi\phi_1 + (1-\pi)\phi_2\right)^2 = \pi(1-\pi)(\phi_1 - \phi_2)^2 \geq 0 \tag{45}$$

Therefore, in large samples, the variance calculated by assuming an AR(1) process will be consistently underestimating the true variance. Equation (45) also shows that the discrepancy depends on both the level of the steady-state probability, and the distance between the regime-wise coefficients. The discrepancy is maximised at $\pi = 0.5$ *ceteris paribus*.