# Optimal Selling Mechanism, Auction Discounts, and Time on Market

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**Abstract:** This article develops a selling with recall framework to examine the optimal selling mechanism problem in real estate market. When sellers can choose between waiting for an optimal number of buyers or auctions (waiting an optimal and fixed time), more risk averse sellers choose auctions and wait a fixed time while less risk averse sellers choose an optimal number of buyers and wait a random time. Positive auction discounts are compensated by decreased risk when time on market is longer than a cut-off time. My study highlights the importance of considering risk aversion, holding cost and downside risk in analyzing real estate selling mechanisms.

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# 1 Introduction

Real estate market is a market with heterogeneous agents trading heterogeneous and illiquid goods. Both buyers and sellers need time to search their deals. From sellers' point of view, it's ideal to sell at a as high as possible price and within a as short as possible time period. In reality sellers face a trade-off between the sale price and time. Time required to sell a property, the so-called time on market (TOM), can be painfully long for some sellers. These sellers have an incentive to sacrifice some return to shorten the TOM. For some other sellers, waiting with uncertainty is not necessarily a painful experience. Patience is a must for high price seekers after all.

A pertinent research question is to determine whether there exists an optimal selling mechanism for all sellers (and all buyers). Adams, Kluger, and Wyatt (1992) compare a "slow Dutch auction" (the seller dynamically decreases his asking prices) and a fixed reserve price strategy. They find that the "slow Dutch auction" is never optimal because its present value of expected profit always shows a positive discount to its counterpart's. Mayer (1995) compares a no reserve, English-style auction to a search sale and also shows positive auction discounts. His model predicts that auction discounts are larger in down market with high vacancies, and in less dense markets. Quan (2002) presents an opposite finding. He shows that prices obtained in auctions are higher than prices obtained in the search market.

There are two common features of these previous studies. First, their models all assume the seller is risk neutral and his aim is to maximize expected revenue. This is consistent with the mainstream of the auction literature in which maximizing expected revenue is the default assumption. However, in real estate market, agents (at least most agents) are not risk neutral. Sellers with different levels of risk aversion may prefer different selling mechanisms. The notion of "optimal mechanism" depends on individuals' preferences. Specifically, if sellers choose to put their houses on auctions, they can largely avoid the TOM uncertainty although there is probability that they may not be able to sell their houses during auctions. These sellers accept lower sale price because their TOM uncertainty is also lower. Second, their models are all based on search models and they then augment search models to accommodate auctions. In their search models sellers cannot recall previous offers.

In this article, I use a "selling with recall" sequential search model to analyze the difference between auctions and search selling mechanisms. Cheng, Lin and Liu (2008) use a similar model to analyze risk premium puzzle in real estate market. In the selling with recall model the seller can recall all or part of previous offers and select among the offers received. The selling with recall model is realistic because earlier buyers may not be able to find a good substitute for the current on-sale property and the seller may not be able to receive any offer above his reserve price. Cheng et al. (2008) analyze a selling mechanism in which the seller waits for an optimal number of buyers. To further justify this selling mechanism, they argue that "Therefore, a rational seller will try to plan for an optimal marketing period. During the planned marketing period he can only expect to receive a finite (and optimal) number of [buyers]." (page 821). There is, however, an important difference between the stopping rule of choosing an optimal time on market (hereafter, SRTM) and the stopping rule of choosing an optimal number of buyers (hereafter, SRNB). If the seller's stopping rule is the SRTM, *ex-ante* TOM and *ex-post* TOM are the same and during the planned TOM he receives a random number of buyers. If the seller's stopping rule is the SRNB, when the optimal buyer arrives the TOM is random.

The SRTM has a duality of search and auction characteristics. On the one hand, the SRTM is a valid search stopping rule (supported by the Cheng et al.'s argument quoted above). Glower, Haurin and Hendershott (1998) find that if sellers plan to move or their jobs have changed, the marketing time of their properties are short. These sellers may have a clear sale deadline but don't necessarily put their properties on auctions. On the other hand, the SRTM can be treated as a private valuation, no reserve, first-price sealed-bid auction in which all remaining buyers send in their offers in sealed envelopes and the seller chooses the highest offered price [1]. In later sections,

[1]The SRTM can also be modified to represent an English auction by assuming the seller chooses the second highest price. It's hard to work out closed-form formulas for an English auction in my model. The simulation algorithm for numerical analysis is provided in Section 4.

I do not distinguish between auctions and the SRTM unless necessary.

My main findings are summarized as follows. First, more risk averse sellers choose the SRTM or auctions while less risk averse sellers choose the SRNB. Both the mean-variance analysis and downside risk analysis show that a unique and universal optimal selling mechanism does not exist. Second, there exists positive auction discounts when two equivalent strategies of the SRTM and the SRNB are compared. Third, auction discounts decrease when market is hot, marketing campaign is efficient, and when buyers' heterogeneity decreases. Auction discounts are compensated by positive risk reductions when TOM is longer than a cut-off time. The cut-off time is shorter when the holding cost is higher. Fourth, when sellers can only choose a fixed TOM, more risk averse sellers with low holding cost choose to wait longer and obtain higher sale price.

I present the details of the model in Section 2. Then, in Section 3, I use the classic mean-variance analysis to analyze and compare alternative selling mechanisms' performance. In Section 4, I analyze and compare downside risks of alternative selling mechanisms. Finally, Section 5 concludes.

## 2 The Model

The selling with recall model described below is similar to that in Cheng et al.'s (2008).

Assume the bid price $X$ is a continuous random variable which has its probability density function $f(x)$ and its cumulative distribution function $F(x)$ on a positive and finite support [A, B]. The distribution of $X$ is known by the seller and is not time dependent. The $n$ independent and identically distributed (i.i.d.) bids are $\{X_1, X_2, ..., X_n\}$. The buyers' arrival follows an exogenous and homogeneous Poisson process with the rate $\lambda$. Adams et al. (1992) and Lin and Vandell (2007) use the same arrival process in their models. There are several well-known properties of a homogeneous Poisson process (see, e.g. Kao 1997). First, the inter-arrival times $t_i$ between the $i-1th$ buyer and the $ith$ buyer are i.i.d. exponential random variables with the mean $1/\lambda$. Second, the arrival time $t$ of the $nth$ buyer follows a gamma distribution with parameters $n$ and

$\lambda$. That is, the probability density function of $t$ is given by

$$g_n(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, \quad t \geq 0 \tag{1}$$

The mean and variance of $g_n(t)$ are $n/\lambda$ and $n/\lambda^2$ respectively. For the SRNB, when the seller chooses an optimal $N^*$, the expect waiting time (holding cost) and its variance are $N^*/\lambda$ ($cN^*/\lambda$) and $N^*/\lambda^2$ ($c^2N^*/\lambda^2$) respectively.

Third, the number of buyers in $[0, t]$ is Poisson distributed with the mean $\lambda t$. That is, for $t \geq 0$

$$Pr(N(t) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \tag{2}$$

Important assumptions of the model are that the bid price distribution $F$ is independent of the bid arrival process governed by parameter $\lambda$ and both are independent of time. The seller has a constant holding cost per unit of time $c$. The same treatment of holding cost has been used in many previous studies (See, e.g. Haurin 1988, Sirmans, Turnbull and Dombrow 1995, Quan 2002). The holding cost $c$ here is an opportunity cost which may be high for the seller. See Section 3.4 for detailed discussion.

The seller chooses the highest price among available offers. The highest price of $n$ offers is $Y_n = max\{X_1, X_2, ..., X_n\}$, $n \geq 1$. Let $F_n(.)$ be the cumulative distribution function of $Y_n$. It's well known in the order statistics literature (e.g. Johnson et al. 1994, page 6-7) that the cumulative distribution function of the highest price is

$$F_n(y) = F(y)^n. \tag{3}$$

The probability density function $f_n(.)$ can be obtained by differentiating $F_n(.)$,

$$f_n(y) = nF(y)^{n-1} f(y). \tag{4}$$

Table 1 lists the cumulative distribution functions, probability density functions, means and variances of $X$ and $Y_n$ when $X$ follows a uniform distribution $U(A, B)$. When bid price distribution is uniform, the mean price of all bids is $\mu_b = (A + B)/2$

and its standard deviation is $\sigma_b = (B - A)/2\sqrt{3}$. . The mean price $\mu_b$ measures the market consensus of the property value. The standard deviation $\sigma_b$ measures the price dispersion which represents buyers' heterogeneity.

***(Insert Table 1 here)***

For the seller who chooses the SRTM, the total benefit of waiting a fixed time period $T$ is

$$K(T) = Y_{N(T)} - cT, \tag{5}$$

For the seller who chooses the SRNB, the total benefit of waiting for $N$ buyers is

$$K(N) = Y_N - cT(N), \tag{6}$$

The seller's utility function is $U(K)$ and his aims to maximize his expected utility. The seller's utility function satisfies $U' > 0$ and $U'' < 0$ (i.e. the seller is risk averse). For the SRTM, each stopping time represents a possible stopping strategy and the seller's decision parameter is $T^*$ that solves the following optimization problem:

$$\max_{T \in (0, +\infty)} E(U(K(T))) \tag{7}$$

For the SRNB, each number of buyers represents a possible stopping strategy and the seller's decision parameter is $N^*$ that solves:

$$\max_{N \in \{1, 2, \ldots, +\infty\}} E(U(K(N))) \tag{8}$$

In some countries (e.g. United States and Singapore) auctions are mainly associated with distress properties. Financially distressed sellers normally have a well specified liquidation date or a limited upper bound of liquidation date. Infinite or even long waiting is not feasible. To analyze the liquidation risk, the selling mechanism has to allow the seller to specify a fixed TOM. A natural way of analyzing the liquidation risk is to treat the liquidation constraint as a pertinent time constraint on the seller's expected utility optimization problem when the seller chooses the SRTM. When the liquidation deadline is $T_L$, the seller's optimization problem is:

$$\max_{T \in (0, T_L)} E(U(K(T))) \tag{9}$$

One special situation is when the seller's unconstrained optimal stopping time $T^*$ is shorter than his liquidation deadline $T_L$. In this situation, Eqs. (7) and (9) are equivalent and the seller's optimal stopping time and maximum expected utility do not change.

During the selling process if no earlier bidder drops his offer, this is the perfect recall situation. When recall is perfect and the optimal number of buyers $N^*$ for the SRNB is determined, the expected TOM is $E(TOM_{N^*}) = \frac{N^*}{\lambda}$ but the TOM is a random variable and can go to infinity. When the optimal TOM $T^*$ for the SRTM is determined, the expected number of buyers is $E(N_{T^*}) = \lambda \times T^*$ but the number of buyers is a random variable. Perfect recall is possible if the TOM is short or the seller is a monopolist. For the SRTM, a short TOM can be chosen so that no buyer drops. When the planned TOM is long, some of the previous buyers may be able to find other appropriate houses. For the SRNB, there is no guarantee of getting perfect recall because the TOM is a random variable.

When some of the earlier buyers drop out and cannot be recalled, this is the partial recall situation. In my model, the mathematical treatment of partial recall is similar to that of Cheng et al.(2008). I assume $\theta$ $(0 \leq \theta \leq 1)$ is the probability of an earlier buyer to stay. Thus $(1 - \theta)$ is the probability of a buyer to exit. When $\theta = 1$, the perfect recall situation recovers. $\theta$ is the same across all buyers thus $\theta$ is also the percentage of all buyers who are still available for recall.

Suppose the seller chooses the SRNB and waits for $N^*$ buyers, on average the remaining number of buyers is $\theta N^*$. The holding cost of the seller is determined by $N^*$ but the sale price is determined by $\theta N^*$. Suppose the seller chooses the SRTM and waits a fixed time $T^*$, on average the remaining number of buyers is $\theta \lambda T^*$. $\lambda$ represents a demand-side factor. The higher the demand, the higher the $\lambda$. $\theta$ represents a supply-side factor. A higher $\theta$ implies fewer buyers find substitutes so the market supply of similar houses is smaller. The product $\theta\lambda$ is the effective arrival rate and represents the offsetting effect of the demand-supply relationship. When demand and supply are both strong (i.e. high $\lambda$ and low $\theta$) the seller benefits from the strong demand but suffers from the strong supply. The effective arrival rate also captures the marketing campaign efficiency. An efficient marketing campaign increases the comparative attractiveness of the house and thus increases the effective arrival rate.

Under close scrutiny, $\theta$ should also be a function of the TOM. Given longer searching time, buyers' probability of finding substitutes increases. Thus the retention rate $\theta$ is not only impacted by the market supply (i.e. an exogenous factor), it's also impacted by the seller's decision (i.e. an endogenous factor). To keep the mathematical analysis tractable, I assume $\theta$ is independent of the TOM [2].

# 3 Mean-Variance Analysis

Markowitz's (1952, 1959) mean-variance analysis lay the foundation of modern portfolio theory. His works explain how risk averse investors make investment choices in a world with uncertainty. In this section, I borrow the classic mean-variance analysis from Markowitz and use it in analyzing selling mechanism choices in real estate market.

[2] Another modeling possibility is to assume that the event of $i$ remaining buyers out of a total $N$ buyers follows a binomial distribution $C_i^N \theta^i (1-\theta)^{N-i}$. For the SRNB, $N = N^*$; for the SRTM, $N$ takes all possible values of arrival numbers. I have conducted a numerical study on this assumption. Numerical study shows that when $N^*$ ($T^*$) is large, treating $\theta N$ as an effective remaining number of buyers for the SRNB (or treating $\theta\lambda$ as an effective arrival rate for the SRTM) provides a good approximation to their binomial distribution counterparts. However, when $N^*$ ($T^*$) is small, the simplified treatments tend to underestimate (overestimate) the variance of sale price for the SRNB (SRTM). Technical details can be obtained from the author upon request. More realistic model may involve advanced stochastic processes (e.g. a birth and death process) to model both the stochastic arrival and the stochastic exit of buyers. It's hard to obtain closed-form formulas in these models however. Computer simulations or numerical methods are the major tools for analyzing the results. In this article, I focus on the simplified mathematical treatments.

## 3.1 Mean and Variance of Total Benefit

When recall is perfect, based on Eq. (6) and results in Table 1 the mean of $K(N)$ is

$$E(K(N)) = E(Y_N) - cE(T(N)) = \frac{NB + A}{N+1} - \frac{cN}{\lambda} \tag{10}$$

The variance of $K(N)$ is

$$Var(K(N)) = Var(Y_N) + c^2 Var(T(N)) = \frac{N(B-A)^2}{(N+1)^2(N+2)} + \frac{c^2 N}{\lambda^2} \tag{11}$$

There is no covariance term in Eq. (11) because $Y_N$ (only depends on bid price distribution) and $T(N)$ (only depends on arrival process) are independent when $N$ is given. For the SRNB, the TOM is a random variable so the holding cost is also a random variable. The variance of the holding cost contributes to the total benefit variance and cannot be ignored by the seller in his decision making.

When recall is partial, the sale price is determined by the effective number of buyers $\theta N$ while the holding cost is still determined by the total arrival number $N$. The mean and variance of the total return $K(N, \theta)$ are:

$$E(K(N,\theta)) = E(Y_{\theta N}) - cE(T(N)) = \frac{\theta NB + A}{\theta N+1} - \frac{cN}{\lambda} \tag{12}$$

and

$$Var(K(N,\theta)) = Var(Y_{\theta N}) + c^2 Var(T(N)) = \frac{\theta N(B-A)^2}{(\theta N+1)^2(\theta N+2)} + \frac{c^2 N}{\lambda^2} \tag{13}$$

It's straightforward to obtain that $\partial E(K(N,\theta))/\partial c < 0$, $\partial E(K(N,\theta))/\partial \lambda > 0$, $\partial Var(K(N,\theta))/\partial c > 0$, and $\partial E(K(N,\theta))/\partial \lambda < 0$. That is, *ceteris paribus*, lower holding cost (higher arrival rate) increases (decreases) the expected value (variance) of total return in both perfect recall and partial recall.

When recall is perfect, based on Eq. (5) the mean of $K(T)$ is

$$\begin{aligned} E(K(T)) &= E(Y_{N(T)}) - cT \\ &= B(1-e^{-\lambda T}) - \frac{B-A}{\lambda T}(1-e^{-\lambda T}-e^{-\lambda T}\lambda T) - cT \end{aligned} \tag{14}$$

The details are provided in Appendix A.

The variance of $K(T)$ is

$$\begin{aligned} Var(K(T)) &= Var(Y_{N(T)} - cT) \\ &= Var(Y_{N(T)}) \\ &= E(Y^2_{N(T)}) - E(Y_{N(T)})^2 \\ &= B^2 - A^2e^{-\lambda T} + \frac{(B-A)^2}{(\lambda T)^2} - (B - Ae^{-\lambda T} + (B-A)\frac{e^{-\lambda T}}{\lambda T})^2 \end{aligned} \tag{15}$$

The details are provided in Appendix B.

When recall is partial, the sale price is determined by the equivalent arrival rate $\theta\lambda$ while the holding cost remains unchanged as $cT$. The mean and variance of the total return $K(T, \theta)$ are:

$$\begin{aligned} E(K(T,\theta)) &= E(Y_{N(T,\theta)}) - cT \\ &= B(1-e^{-\theta\lambda T}) - \frac{B-A}{\theta\lambda T}(1-e^{-\theta\lambda T}-e^{-\theta\lambda T}\theta\lambda T) - cT \end{aligned} \tag{16}$$

and

$$\begin{aligned} Var(K(T,\theta)) &= Var(Y_{N(T,\theta)} - cT) \\ &= Var(Y_{N(T,\theta)}) \\ &= B^2 - A^2e^{-\theta\lambda T} + \frac{(B-A)^2}{(\theta\lambda T)^2} - (B - Ae^{-\theta\lambda T} + (B-A)\frac{e^{-\theta\lambda T}}{\theta\lambda T})^2 \end{aligned} \tag{17}$$

For the SRTM, the fixed holding cost has no contribution in the variance. So the variance of total benefit is the same as the variance of the sale price.

Sale price is of interest for empirical studies because the holding cost is unobservable in most cases. From Eqs. (16) and (17), the mean and variance of the sale price are:

$$E(Y_{N(T,\theta)}) = B(1-e^{-\theta\lambda T}) - \frac{B-A}{\theta\lambda T}(1-e^{-\theta\lambda T}-e^{-\theta\lambda T}\theta\lambda T) \tag{18}$$

and

$$Var(Y_{N(T,\theta)}) = B^2 - A^2e^{-\theta\lambda T} + \frac{(B-A)^2}{(\theta\lambda T)^2} - (B - Ae^{-\theta\lambda T} + (B-A)\frac{e^{-\theta\lambda T}}{\theta\lambda T})^2. \tag{19}$$

$\theta$, $\lambda$ and $T$ play symmetrical roles in the mean and variance of the sale price. A direct implication of this symmetry is that the seller has to wait more time to maintain the mean and variance of sale price when the effective arrival rate $\theta\lambda$ goes down.

Taking the first derivative of $E(Y_{N(T,\theta)})$ with respect to $T$,

$$\frac{\partial E(Y_{N(T)})}{\partial T} = A\theta\lambda e^{-\theta\lambda T} + \frac{\theta\lambda(1-e^{-\theta\lambda T}-e^{-\theta\lambda T}\theta\lambda T)}{(\theta\lambda T)^2}(B-A) \tag{20}$$

It's obvious to see that the first part of Eq. (20), $A\theta\lambda e^{-\theta\lambda T}$, is positive. The second part $\frac{\theta\lambda(1-e^{-\theta\lambda T}-e^{-\theta\lambda T}\theta\lambda T)}{(\theta\lambda T)^2}(B-A)$ is also positive. The sum $\sum_{n=1}^{\infty} e^{-\theta\lambda T}\frac{(\theta\lambda T)^n}{n!} = 1-e^{-\theta\lambda T} < 1$ and $e^{-\theta\lambda T}$, $e^{-\theta\lambda T}\theta\lambda T$ are the first two terms in the sum and all terms in the sum are positive, so $(1-e^{-\theta\lambda T}-e^{-\theta\lambda T}\theta\lambda T) > 0$. Overall, $\frac{\partial E(Y_{N(T)})}{\partial T} > 0$. The (realized) TOM (or $\theta$, or $\lambda$ by symmetry) is positively related to the expected sale price and the relationship is nonlinear. This result is consistent with the theoretical finding of Cheng et al.(2008). The empirical implication of this result is that linear econometric models on TOM-price relationship are misspecified.

## 3.2 Opportunity Sets and Efficient Sets

In this section, I conduct analysis and comparison of the stopping rules SRTM and SRNB based on the classic mean-variance portfolio theory. In my analysis, I assume that the seller has his mean-variance utility on total benefit $K$ (including the holding cost) [3]. Hence, the objective of the seller is to maximize $E(K) - \gamma Var(K)$, $\gamma > 0$. The constant $\gamma$ measures the degree of risk aversion: the larger the $\gamma$ is, the more risk averse the seller is. To compare the opportunities provided by alternative stopping rules, I borrow the concepts of the opportunity set and the efficient set from the mean-variance portfolio theory and redefine them in the selling mechanism context.

**Definition 1** The opportunity set represents the set of all possible stopping strategies that could be obtained by a selling mechanism. The efficient set represents the set of stopping strategies that offer maximum expected total benefit for varying levels of risk and minimum risk for varying levels of expected total benefit that could be obtained by a selling mechanism.

The risk averse seller will choose his optimal stopping strategy (an optimal stopping time or stopping number) from the efficient set according to his risk aversion. The study in this section focuses on the comparison of the opportunity sets and the efficient sets when the seller faces two alternative selling mechanisms. If the SRTM and the SRNB are equivalent, their opportunity sets should be the same. If this is the case, introducing the SRTM to the seller who previously chooses the SRNB does not increase his set of possible choices and the SRTM is redundant. Another layer of analysis is to compare whether the efficient set generated by the SRTM (SRNB) is dominated by the efficient set generated by the SRNB (SRTM). If either of these situations happens, the optimality of one selling mechanism exists. If the dominance of either efficient set does not exist, then both the SRTM and the SRNB have their own merits and both should be considered as viable selling mechanisms.

Using the closed-form mean and variance formulas obtained in Section 3.1, Figure 1 shows the opportunity sets in the classic mean - standard deviation ($E$ - $\sigma$) diagram. Panel (a) shows the opportunity sets when recall is perfect (i.e. $\theta = 1$). Panel (b) shows

[3]Cheng et al. (2008) use the mean-variance utility only on sale price (excluding the holding cost). It seems not convincing because the variance of the holding cost is non-zero for the SRNB and should be taken into account when the seller tries to determine his optimal stopping strategy.

the opportunity sets when recall is partial, I choose the parameter $\theta = 0.25$ (the same parameter in the Cheng et al.'s numerical study). Other parameters are:

1) $\lambda = 10$ per month (the same as that of Cheng et al.'s numerical study);

2) $B = 100,000$, $A = 75,000$;

3) $c = 3,000$.

For the SRNB, the opportunity set is calculated on $N = \{5, ..., 60\}$. For the SRTM, the opportunity set is calculated on $T = N/\lambda = \{0.5, ..., 6\}$. The opportunity sets of the two selling mechanisms are both curves [4]. The rightmost points of the opportunity sets are the first pair $\{N = 5, T = 0.5\}$. For the SRNB, it represents that the seller sells his property when the fifth bid is received; for the SRTM, it represents that the seller waits 0.5 month and then sells the property. The ending points of both opportunity sets are intended choices. According to the National Association of Realtors, the average TOM for the U.S. residential market was about six months during the period of 1989-2004. It's of empirical interest to compare the total benefit pair $K(T = 6)$ and $K(N = \lambda T = 60)$ (points labeled in Figure 1 by star and diamond respectively). Actually, the opportunity sets shown in Figure 1 are parts of the full sets but they are enough to reveal the important characteristics of the full opportunity sets.

In both panels, $K(N = 5)$ dominates $K(T = 0.5)$ by showing higher expected total benefit and lower standard deviation (risk). Both points however are not on the efficient sets. The efficient set of the SRNB is a curve between two points of the opportunity set. One point is the stopping strategy which gives the highest expected total benefit. This point represents the maximum mean stopping strategy. The other point is the stopping strategy which gives the lowest risk. This point represents the minimum variance stopping strategy.

When $\theta = 1$, using Eq. (10), the maximum mean stopping strategy point of the SRNB can be obtained analytically by solving the following equation

$$\frac{\partial E(K(N))}{\partial N} = \frac{B - A}{(N + 1)^2} - c/\lambda = 0 \tag{21}$$

[4]Strictly speaking, the opportunity set of the SRNB is comprised of disjointed points.

The optimal number of buyers of the maximum mean stopping strategy is $N^*_{MMS} = \sqrt{(B-A)\lambda/c} - 1$. Plugging in the parameters, $N^*_{MMS} = \sqrt{(100,000-75,000)10/3,000} - 1 \approx 8$.

Using Eq. (11), the the minimum variance stopping strategy point of the SRNB can be obtained by solving the following equation

$$\frac{\partial Var(K(N))}{\partial N} = -2(B-A)^2 \frac{N^2+N-1}{(N+1)^3(N+2)^2} + c^2/\lambda^2 = 0 \tag{22}$$

Plugging in the parameters and solving the equation by numerical root-finding method, the number of buyers of the minimum variance stopping strategy $N^*_{MVS} \approx 22$. The efficient set of SRNB when $\theta = 1$ is the curve between $\{E = 94.82, \sigma = 2.63\}$[5] (when $N = 8$) and $\{E = 92.31, \sigma = 1.75\}$ (when $N = 22$).

The efficient set of the SRTM is also a curve between two points. One point represents the maximum mean stopping strategy and the other one represents the minimum variance stopping strategy for which the TOM is $T = 6$.

When $\theta = 1$ ,using Eq. (14), the maximum mean stopping strategy point of the SRTM can be obtained by solving the following equation:

$$\frac{\partial E(K(T))}{\partial T} = A\lambda e^{-\lambda T} + \frac{\lambda(1 - e^{-\lambda T} - e^{-\lambda T}\lambda T)}{(\lambda T)^2}(B-A) - c = 0 \tag{23}$$

Plugging in the parameters and solving the equation by numerical method, the TOM of the maximum mean stopping strategy $T^*_{MMS} \approx 0.9$. The efficient set of the SRTM is the curve between $\{E = 94.45, \sigma = 2.86\}$ (when $T = 0.9$) and $\{E = 81.58, \sigma = 0.42\}$ (when $T = 6$). There exists a global minimum variance point for the SRTM. Using Eq. (15), $\lim_{T\to+\infty} Var(K(T)) = 0$ so $T^*_{MVS} = \infty$. Points representing very short TOM are not on the efficient set of the SRTM. The non-zero optimal TOM is an intrinsic feature of the SRTM. The seller needs time to market his property and to attract enough buyers. When $\theta < 1$, the efficient sets of the SRNB and the SRTM can be determined by similar approaches.

[5]Numbers are in thousands.

It's interesting that in both panels the point $K(T=6)$ is not only on the efficient set of the SRTM but also on the combined efficient set generated by combining the opportunity sets of the SRTM and the SRNB. The point $K(N=60)$, however, is not on the combined efficient set in both panels. In Panel (a), the point $K(N=60)$ is not even on the efficient set of the SRNB. Thus when recall is perfect, waiting for $N=60$ buyers is not an optimal strategy for any risk averse seller if he only chooses stopping strategies of the SRNB. When recall is partial, waiting for $N=60$ buyers is not an optimal strategy for any risk averse seller if he chooses stopping strategies from both the SRTM and the SRNB.

This result is robust under different parameter settings and it suggests that the average 6 months TOM observed in the U.S. residential market may be better explained by the SRTM rather than by the SRNB. Of course the interpretation of this result should not go farther than the mean-variance utility framework.

Overall, Figure 1 demonstrates a selling mechanism to selling mechanism comparison between the SRTM and SRNB. The efficient set of combining the SRTM and the SRNB is significantly different from the efficient set generated by the SRTM or the SRNB alone. Both selling mechanisms contribute part of the combined efficient set. If the seller is less (more) risk averse, he may choose the SRNB (SRTM) to obtain higher (lower) expected total benefit with higher (lower) associated risk. A direct theoretical implication of this result is that auctions as a selling mechanism that largely eliminates the TOM uncertainty are preferred by more risk averse sellers. The result is consistent with the empirical facts that governments, banks and financially distressed sellers tend to use auctions as their selling mechanism. Another important theoretical implication of my result is that a unique and universal optimal selling mechanism does not exist in real estate market. It's supported by empirical evidence that both search and auction markets are used by sellers. Another implication is that the definition of auction discount (regardless it's positive, zero or negative) needs to be reexamined. When risk aversion leads different sellers to choose different stopping strategies within one selling mechanism, defining the auction discount on a selling mechanism-to-selling mechanism basis is not any more valid. If two stopping strategies are randomly chosen from different selling mechanisms, the difference between their expected total benefit could be positive, zero or negative. An related empirical implication is that researchers should control more factors than hedonic factors when they estimate auction discounts from data.

## 3.3 Auction Discounts and Risk Reductions

My model provides a new way to interpret and analyze auction discounts when risk is incorporated. When the seller is able to choose different stopping strategies in one selling mechanism, the auction discount should be defined on the stopping strategy level rather than on the selling mechanism level. Auction discounts are not necessarily an economic handicap if the associated risk is reduced. To quantify and analyze auction discounts and risk reductions, I define the waiting equivalent stopping strategy and waiting equivalent TOM as follows.

**Definition 2** For each stopping strategy $N$ (waiting for $N$ buyers) of the SRNB, its waiting equivalent stopping strategy is the stopping strategy of the SRTM which satisfies $T_{we}(N) = N/\lambda$ (waiting a fixed time $T_{we}(N)$). $T_{we}$ is the waiting equivalent TOM.

The average waiting time of waiting for $\theta N$ buyers is $\theta N/\lambda = \theta T_{we}(N)$. As shown in Section 3.2, there are many feasible strategies in one selling mechanism. The definition of waiting equivalent TOM provides a foundation of strategy-to-strategy comparison between the comparable strategies of the SRNB and the SRTM.

After defining the waiting equivalent strategy and the waiting equivalent TOM, I define auction discount and risk reduction as follows.

**Definition 3** Auction discount is the difference between the expected total benefit $E(K(N, \theta))$ of one stopping strategy of the SRNB and the expected total benefit

$E(K(T_{we}(N), \theta))$ of its waiting equivalent stopping strategy of the SRTM: $AD(N, \theta) = E(K(N, \theta)) - E(K(T_{we}(N), \theta))$.

From Definition 2, $cN/\lambda = cT_{we}$. Plugging this equality into Eqs. (12) and (16), $AD(N, \theta) = E(Y_{\theta N}) - E(Y_{T_{we}, \theta})$. So the auction discount $AD(N, \theta)$ is also the difference between expected sale prices of two comparable strategies.

**Definition 4** Risk reduction is the difference between the standard deviation $\sigma(K(N, \theta))$ of one stopping strategy of the SRNB and the standard deviation $\sigma(K(T_{we}(N), \theta))$ of its waiting equivalent stopping strategy of the SRTM : $RR(N, \theta) = \sigma(K(N, \theta)) - \sigma(K(T_{we}(N), \theta))$.

Theorem 1 summarizes several important conclusions on auction discounts and risk reductions.

**Theorem 1** (1) Auction discounts are always positive. (2) Auction discounts decrease when $\theta$ ($\lambda$ or $T_{we}$) increases and $\theta N \geq 1$ (at least one buyer remains for recall). (3) Auction discounts decrease when buyers' heterogeneity decreases and $\theta N \geq 2$ (at least two buyers remain for recall). (4) Risk reductions increase when holding cost $c$ increases.

Details of the proof are provided in Appendix D.

The first conclusion, positive auction discounts, is consistent with the results obtained by Adams et al. (1992) and Mayer (1995). The second conclusion, auction discounts decrease when $\theta$ increases (i.e. when market is hot and marketing campaign is efficient), is consistent with the result obtained by Mayer (1995). The third conclusion, auction discounts decrease when buyers' heterogeneity decreases, is consistent to the result obtained by Mayer (1995). Buyers' heterogeneity is closely related to atypical houses. When a house has unusual features, the distribution of offers tend to have a larger variance. Mayer (1995) shows that on-sale houses that are more homogeneous have a smaller mismatch cost and thus a smaller auction discount. My conclusion is also loosely consistent with the results obtained by Haurin (1988) and Glower et al. (1998). They show that when seller's house is atypical, the TOM is longer and the expected sale price is higher. The economic intuition of the third conclusion is that when buyers' private valuations are similar ($\sigma_b$ is small), the value of flexible waiting time decreases. The fourth conclusion is new. None of the previous studies check the risk reduction associated to the auction discount. The variance of holding cost enters the variance of the SRNB but not the variance of the SRTM. This is because that the seller who chooses the SRNB faces TOM uncertainty. The higher the holding cost, the higher the contribution of the holding cost in the variance of the SRNB which contributes to risk reductions' increase.

***(Insert Figure 2 here)***

Figure 2 illustrates the relationship between auction discounts, $\theta$, risk reductions and holding cost $c$ at different lengths of waiting equivalent TOM. The parameters $\{\lambda, B, A\}$ chosen are the same as those in Section 3.2. In both panels, auction discounts are always positive. Auction discounts decrease when the waiting equivalent TOM is longer. Panel (a) shows that auction discounts increases when $\theta$ decreases. Panel (b) shows the relationship between auction discounts and risk reductions when recall is perfect ($\theta = 1$). The risk reductions increase when the waiting equivalent TOM is longer and holding cost is higher. For each risk reductions curve, there exists a cut-off time when risk reduction is zero. The cut-off time shortens when the holding cost increases. Auction discounts are compensated by risk reductions when the waiting equivalent TOM is longer than the cut-off time. When the waiting equivalent TOM is shorter than the cut-off time, individual SRNB strategies dominate their SRTM waiting equivalent counterparts by showing higher returns and lower risks. When the waiting equivalent TOM is longer than the cut-off TOM, the dominance disappears. The seller will choose his optimal strategy based on his risk aversion.

### 3.4 Holding Cost, Risk Aversion, and TOM

The holding cost $c > 0$ is a crucial assumption in the selling with recall model. If the holding cost is zero, the seller is always better off by waiting for the next bidder (or waiting more time).

Using the cumulative distribution function derived in Appendix C, I show two dominance results for both the SRNB and the SRTM when the holding cost is zero. For the SRNB, $Pr(K(N) \leq k) = F_n(k)$. Since $0 \leq F(k) \leq 1$ for all $k$ and $0 < F(k) < 1$ for some $k$, $F_n(k) \leq F_{n-1}(k)$ for all $k$ and $F_n(k) < F_{n-1}(k)$ for some $k$. Thus, waiting for $n$ buyers dominates waiting for $n-1$ buyers by the first-degree stochastic dominance. Similar result can be obtained for the SRTM. The cumulative distribution function $Pr(K(T) \leq k) = \sum_{n=0}^{\infty} F_n(k) e^{-\lambda T} \frac{(\lambda T)^n}{n!}$ which is a weighted average of $F_n(k)$ [6]. When $T$ is larger, more weights are put on larger $n'$s and smaller values of $F_n(k)$. Thus $Pr(K(T_1) \leq k) \leq Pr(K(T_2) \leq k)$ when $T_1 > T_2$. The equations hold if and only if $k = A$ or $B$. Thus waiting $T_1$ dominates waiting $T_2$ by the first-degree stochastic dominance. In summary, for any investor with utility function $U$ satisfying $U' > 0$ (regardless he is risk neutral, risk averse or risk loving), he always prefers waiting for more buyers (more time) when the holding cost is zero. This result is independent of the bid price distribution.

When the holding cost is non-zero, the seller considers the trade-off of the benefit of waiting and the cost of waiting. The holding cost is an opportunity cost and the cost can be large. Quan (2002) quotes the *Wall Street Journal* report which shows the annual holding cost for commercial property can be as high as 20% of appraised value. For many households, residential properties have locked in most of their equity (Cocco 2004, Hu 2005). When they sell their houses, a significant amount of equity is available to consume or invest.

Another important opportunity cost concept in real estate market is the user cost (cost of house ownership). Himmelberg, Mayer and Sinai (2005) use a formula to calculate the user cost in the U.S. In their formula they include six components of

[6] Define $F_0(k) = (\frac{k-A}{B-A})^0 = 1$.

the user cost: 1) cost of foregone interest that the homeowner could have earned by investing in something other than a house; 2) cost of property taxes; 3) tax deductibility of mortgage interest and property taxes; 4) maintenance costs; 5) expected capital gain (or loss) of the house; 6) an additional risk premium to compensate homeowners for the higher risk of owning versus renting. They show that the user cost can vary across cities and within cities over time. For example, they show that the average user cost in San Jose is 3.3% and 7.1% in Pittsburgh.

The holding cost is in general higher than the user cost. There are several reasons to support the argument. First, when estimating the holding cost, the $5th$ component (expected capital gain (or loss) of the house) should be excluded to avoid double accounting. Because the potential gain/loss is already included in the expected sale price. In general, houses price grows rather than declines so excluding this component increases the opportunity cost. Second, other components in the user cost are still valid components in the holding cost because the owner still occupies the house before the sale happens. Sirmans et al. (1995) provide evidence that vacant houses have higher holding cost than those occupied by owners. Third, marketing expenses should be included in the holding cost.

***(Insert Figure 3 here)***

Figure 3 illustrates the impacts of the seller's holding cost and risk aversion on his optimal TOM if he can only choose the SRTM. The parameters $\{A, B, c, \lambda\}$ chosen are the same as those in Section 3.2. Parameter $\gamma = \{0.1, 1, 10\}$ represents different degrees of risk aversion. Figure 3 shows that the seller's optimal TOM decreases when his holding cost increases. The result is consist with a number of previous studies (Sirmans et al. 1995, Mayer 1995, Cheng et al. 2008). Figure 3 also shows that the seller's optimal TOM increases when the seller is more risk averse.

The findings in this section have several empirical implications. Empirical studies show that real estate returns had extremely low volatility and extremely high risk-adjusted returns comparing to other asset classes. This is the so-called risk premium puzzle. Lin and Vandell (2007) and Cheng et al. (2008) suggest that one promising approach to understand the puzzle is to adjust the observed sale price and volatility in illiquid market to their full liquid market counterparts. They show that after adjustments, real estate return is lower and its volatility is higher. Opportunity costs also provide a way to help explain the risk premium puzzle in real estate market. During the occupying period of a property, the owner faces the user cost. When the owner puts the property on sale, he faces the holding cost. These costs can be high and the total benefit of selling a property can be dramatically lower than that suggested by the sale prices. When TOM is uncertain, the uncertainty of holding cost increases the volatility of total benefit which is not captured by sale price volatility. Overall, observed sale prices tend to overestimate real estate return and underestimate its volatility.

Using the data from Boston condominium market of the early 1990's, Genesove and Mayer (1997) find that the seller of a property with a high loan-to-value ratio has a longer TOM and receives a higher price than a seller with less debt. They argue that an equity constrained seller has to choose a higher asking price to cover the down payment for his next property purchase. [7] According to my model, there are at least two other ways to explain Genesove and Mayer's empirical finding. First, sellers with high loan-to-value ratio properties are more risk averse. This argument is by no means far away from economic intuition. Second, sellers holding high loan-to-value ratio properties have low holding cost. I use a brief thought experiment to support the low holding cost argument. Suppose two properties A and B have the same market value. Property A (B) has a high (low) loan-to-value ratio. The weighted average cost of capitals (WACCs) of these two properties are different. Property A has lower WACC while property B has higher WACC. Thus the equity constrained sellers tend to have lower holding cost. The above thought experiment simplifies the calculation of the holding cost which should include several other components. Nonetheless, it provides a straightforward way to show the difference. To conclude, more risk averse seller with lower holding cost will wait longer and obtain higher sale price.

[7] Genesove and Mayer (2001) further find that loss aversion contributes more than equity constraint in explaining the seller's behavior. I will discuss downside risk in Section 4.

# 4 Downside Risk Analysis

In this section, I extend the mean-variance analysis by analyzing the downside risk the seller faces. The mean-variance utilities are widely used to model risk averse agents in the finance and economics literature. They provide approximations of the utility functions up to the second order expansion. However, the use of variance as a risk measure is valid only if the seller's utility function is quadratic or the total benefit distributions differ by scale and location parameters (e.g. normal distributions). The seller's utility function may not be quadratic. Genesove and Mayer (2001) show that sellers of residential properties have strong loss aversion. When sellers have strong loss aversion, they focus their attention on downside risk. The seller's utility function may even change over time. Albrechet, Anderson, Smith and Vroman (2007) show a search model in which both buyers and sellers begin with relaxed states and move to desperate states if there is no match. Non-normality of total benefits is clear to see when the upper and lower bounds of total benefit of selling mechanisms are considered.

## 4.1 Upper and Lower Bounds of Total Benefit

For the SRNB, the seller's waiting time can go to infinity when the optimal bidder arrives. The highest price the seller can get is $B$ which is finite. Thus the upper bound of the total benefit is $B$ and the lower bound is $-\infty$. The seller faces a potential cost blow-out. For the SRNB, $K(N, \theta)$ is on a continuous support $[-\infty, B]$.

For the SRTM, the worst scenario happens when no bidder arrives during the planned TOM. Auctions are unsuccessful in this scenario. The total benefit is negative and is equal to the holding cost $-cT$. $-cT$ is the lower bound of $K(T, \theta)$. When time reaches the planned TOM, the probability of getting no bidder is $e^{-\theta\lambda T}$ which is non-zero. The probability of successful auctions increases when the market is good and the seller is efficient in attracting buyers. Ong, Lusht and Mak (2005) show empirical evidence that more bidder turnout (a proxy for the number of buyers at an auction), efficient auction houses and good market condition contribute to successful auction outcomes. When one or more buyers come, the maximum price the seller can get is $B$, the upper bound of $K$ is $B-cT$ and $K(T,\theta)$ is on a continuous support $[A-cT, B-cT]$. Overall, $K(T,\theta)$ is on the union $[-cT] \cup [A-cT, B-cT]$. The SRTM has larger lower bound and smaller upper bound comparing to the SRNB.

By analyzing the upper and lower bounds of total benefits it's clear that the total benefits for both the SRNB and the SRTM are not normally distributed. Downside risks need to be considered when the seller makes his selling mechanism choices.

## 4.2 Certainty Equivalent TOM

How to compare the downside risks of the SRTM and the SRNB remains an issue. As shown in Theorem 1, the expected total benefit $E(K(N,\theta)) > E(K(T_{we}(N),\theta))$. However, it's desirable to compare the downside risks on pairs of $\{N, T\}$ when $E(K(N,\theta)) = E(K(T,\theta))$. The seller may want to minimize his downside risk given certain level of expected total benefit. This requirement is partially achieved by defining the certainty equivalent stopping strategy and certainty equivalent TOM as follows.

**Definition 5** For each stopping strategy $N$ in SRNB, its certainty equivalent stopping strategy is the strategy of the SRTM which satisfies $E(K(N,\theta)) = E(K(T_{ce}(N),\theta))$. $T_{ce}(N)$ is the certainty equivalent TOM.

The certainty equivalent TOM can be numerically obtained by solving the $T$ from the following non-linear equation: $\frac{\theta NB+A}{\theta N+1} - \frac{cN}{\lambda} = B(1-e^{-\theta\lambda T}) - \frac{B-A}{\theta\lambda T}(1-e^{-\theta\lambda T} - e^{-\theta\lambda T}\theta\lambda T) - cT$. There are two properties of the certainty equivalent TOM. First, for some $N$, $T_{ce}(N)$ does not exist. Second, for some $N$, there exist two possible values of $T_{ce}(N)$. Both situations are clear to see in Figure 1. When there are more than one certainty equivalent TOM, I choose the longer one in my numerical study below.

## 4.3 Value at Risk and Expected Shortfall

I use two widely used measures, value at risk (VaR) and expected shortfall (ES), to measure the downside risks of comparable strategies in alternative selling mechanisms. VaR and ES are widely used in the finance literature and financial industry (see, e.g. Duffie and Pan 1997, Frey and McNeil 2002). VaR is recommended by the Basel Committee for Banking Regulation to establishing a bank's capital adequacy requirements. Despite these two measures' large impacts on finance, few real estate studies adopt them to measure downside risks [8].

The definitions of value at risk and expected shortfall are as follows.

**Definition 6** Value at Risk (VaR): Given a confidence level $\alpha$, the VaR at confidence level $\alpha$ is the smallest value $l$ such that the probability that the loss $L$ exceeds $l$ is no larger than $1-\alpha$. In other words,

$$VaR_\alpha = inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\} \tag{24}$$

Statistically, VaR is the lower $\alpha$-quantile in terms of a generalized inverse of the distribution function $F_L$.

**Definition 7** Expected Shortfall (ES): For a continuous loss distribution with $\int_{\mathbb{R}} |l| dF_L(l) < \infty$, the $ES_\alpha$ at confidence level $\alpha \in (0,1)$ for loss $L$ is:

$$ES_\alpha = E(L|L \geq VaR_\alpha) = \frac{\int_{VaR_\alpha}^{\infty} l dF_L(l)}{1-\alpha} \tag{25}$$

In my study, I calculate VaR and ES with respect to $K$ and its distribution. Higher values of VaR and ES imply smaller downside risk. One way to obtain risk measures VaR and ES numerically is through numerical integration. The cumulative distribution functions and probability density functions of $K(T,\theta)$ and $K(N,\theta)$ have explicit forms and can be used for numerical integration. Details are provided in Appendix C. Another way is through the Monte Carlo simulation. I use this approach in my numerical study. The simulation algorithms of homogeneous Poisson arrival and arrival time are

[8] An exception is Gan and Hill (2009). They adapt VaR to measure housing affordability.

well known (e.g. Kao 1997, page 52). I use the following augmented algorithms to simulate $K$:

**Algorithm 1 - SRTM, simulation of** $K(T, \theta)$

Set $A, B, c, T, \theta, \lambda$.

**for** i=1, 2, ..., M

Generate uniform random variable $U_j \sim U(0,1)$, $j = 1, 2, ...$, until the condition $U_1 ... U_n \geq e^{-\theta\lambda T}$ is violated for the first time. Let $U_N$ be the last uniform random variable obtained, the simulated $N(T)$ is then given by $N - 1$.

Generate $N(T)$ uniform random variables $Y_j \sim U(A, B)$, $j = 1, ..., N(T)$ and choose the highest value $Y_{max}$ of them [9].

$K(i) = Y_{max} - cT$

**end (for)**

**Algorithm 2 - SRNB, simulation of** $K(N, \theta)$

Set $A, B, c, N, \theta, \lambda$.

**for** i=1, 2, ..., M

Generate $U \sim U(0,1)$.

$T = -(1/\lambda) log(U)$.

Generate $N$ uniform random variables $Y_i \sim U(A, B)$, $j = 1, ..., N$ and choose the highest value $Y_{max}$ of them.

$K(i) = Y_{max} - cT$

**end (for)**

I simulate $M = 100,000$ independent observations of $K$ and calculate VaR and ES based on simulated observations. The program is written in MATLAB. The parameters $\{\lambda, B, A, c\}$ chosen are the same as those in Section 3.2. I examine both the perfect recall ($\theta = 1$) and the partial recall ($\theta = 0.25$). Confidence level $\alpha = 0.99$. I choose

[9] Results of an English auction can be obtained by choosing the second highest value in this step.

$N = \{8, 16, 32, 64\}$ so that $\theta N$ are integers.

Table 2 listed the simulation results. For the SRNB, when recall is perfect, $N = 8$ gives the highest mean. $N = 16$ gives the smallest downside risk and the smallest standard deviation. $N = 16$ gives a good example in which the downside risk is high when the expected total benefit is high and volatility is low. The $VaR_{0.99}$ is 88.26 (thousands) which represents a 6% discount comparing to the expected total benefit. When recall is partial, $N = 16$ gives the highest mean, $N = 32$ gives the smallest downside risk, and $N = 64$ gives the smallest standard deviation. For the SRTM, when recall is perfect, $T_{ce} = 1.55$ (months) gives the highest mean and the smallest downside risk, $T_{ce} = 6.40$ (months) gives the smallest standard deviation. When recall is partial, downside risk rankings are inconsistently given by VaR and ES. VaR suggests that $T_{ce} = 3$ (months) has smaller downside risk while ES suggests that $T_{ce} = 6.27$ (months) has smaller downside risk [10].

When conducting cross selling mechanism comparison, it's obvious in Table 2 that risk neutral, risk averse and loss averse sellers will choose different optimal stopping strategies of different selling mechanisms. Two strategies with the same expected total return may be preferred by different sellers. For example, when recall is perfect, $N = 16$ and $T_{ce} = 1.55$ (months) give the same expected total benefit. Mean-variance utility sellers will choose $T_{ce} = 1.55$ (months) because of its lower standard deviation. Downside risk minimizers however will choose $N = 16$ instead because of its lower downside risks. Again, a unique and universal optimal selling mechanism for all sellers does not exist.

The results in this section emphasize the importance of downside risks. The downside risks of selling properties are closely related to TOM uncertainty. The great recession of 2007/2009 re-emphasizes the importance of quantifying and measuring downside risks in real estate market.

[10] Since the seminal work of Artzner, Delbaen, Eber and Heath (1999) it is now well known that VaR is not a coherent risk measure and ES is a coherent risk measure.

# 5 Conclusion

Conventional wisdom tells us that modern finance theory is a branch of applied microeconomics. In this article, I show that the modern finance theory sheds light on a conventional microeconomic problem. Mean-variance analysis and downside risk analysis help advance the understanding of optimal selling mechanism problem in real estate market. I show that risk aversion, holding cost and downside risk are important factors influencing sellers' selling mechanism choices. Different sellers may choose different optimal selling mechanism so the notion of "optimal mechanism" is user dependent. In reality English auctions dominate the real estate market (Mayer 1995, Ong 2006). Further numerical study can be done by using the algorithm provided in Section 4. In my study, I use the selling with recall framework. It's also of interest to analyze a traditional selling without recall model. How to adapt downside risk measures so that they can be used in real estate market deserves more study. Much remains to be done in cross-country comparison of selling mechanisms. In New Zealand and Australia, auctions are used more frequently than in the United States. This difference may be driven by the micro-factors analyzed in this article.

## Appendix A

I prove several auxiliary results first.

$$\sum_{n=1}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} = 1 - e^{-\lambda T} \tag{26}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{n+1} e^{-\lambda T} \frac{(\lambda T)^n}{n!} \\
= \quad & \frac{1}{\lambda T} \sum_{n=1}^{\infty} e^{-\lambda T} \frac{(\lambda T)^{n+1}}{(n+1)!} \\
= \quad & \frac{1}{\lambda T} \sum_{m=2}^{\infty} e^{-\lambda T} \frac{(\lambda T)^m}{m!} \\
= \quad & \frac{1}{\lambda T} (1 - e^{-\lambda T} - \lambda T e^{-\lambda T})
\end{aligned} \tag{27}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{n+2} e^{-\lambda T} \frac{(\lambda T)^n}{n!} \\
= \quad & \sum_{n=1}^{\infty} \frac{n+1}{(\lambda T)^2} e^{-\lambda T} \frac{(\lambda T)^{(n+2)}}{(n+2)!} \\
= \quad & \frac{1}{(\lambda T)^2} \sum_{m=3}^{\infty} (m-1) e^{-\lambda T} \frac{(\lambda T)^m}{m!} \\
= \quad & \frac{1}{(\lambda T)^2} (\lambda T - 1 + e^{-\lambda T} - e^{-\lambda T} (\lambda T)^2/2)
\end{aligned} \tag{28}$$

Proof of $E(K(T))$. From Eq. (14),

$$\begin{aligned}
E(Y_{N(T)}) &= E(E(Y_N(T)|N(T)=n)) \\
&= \sum_{n=1}^{\infty} \frac{nB+A}{n+1} e^{-\lambda T} \frac{(\lambda T)^n}{n!} \\
&= \sum_{n=1}^{\infty} \left(B - \frac{B-A}{n+1}\right) e^{-\lambda T} \frac{(\lambda T)^n}{n!} \\
&= B\sum_{n=1}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} - (B-A)\sum_{n=1}^{\infty} \frac{1}{n+1} e^{-\lambda T} \frac{(\lambda T)^n}{n!} \\
&= B(1-e^{-\lambda T}) - \frac{B-A}{\lambda T}(1-e^{-\lambda T} - e^{-\lambda T}\lambda T)
\end{aligned}$$

Then $E(K(T)) = E(Y_{N(T)}) - cT = B(1-e^{-\lambda T}) - \frac{B-A}{\lambda T}(1-e^{-\lambda T} - e^{-\lambda T}\lambda T) - cT$.

## Appendix B

Proof of $Var(K(T))$.

$$E(Y^2|N(T)=n;n=0)=0$$

$$\begin{aligned}
E(Y^2|N(T)=n;n\geq 1) &= \int_A^B y^2 f_n(y)dy \\
&= \int_A^B y^2 dF_n(y) \\
&= y^2F_n(y)|_A^B - \int_A^B F_n(y)dy^2 \\
&= B^2 - \int_A^B 2yF_n(y)dy \\
&= B^2 - 2\int_A^B y\left(\frac{y-A}{B-A}\right)^n dy \\
&= B^2 - 2\int_A^B \frac{(y-A)^{n+1}}{(B-A)^n}dy - 2\int_A^B A\left(\frac{y-A}{B-A}\right)^n dy \\
&= B^2 - 2\int_0^{B-A} \frac{z^{n+1}}{(B-A)^n}dz - 2A\int_0^{B-A}\frac{z^n}{(B-A)^n}dz \\
&= B^2 - 2(B-A)^2\frac{1}{n+2} - 2A(B-A)\frac{1}{n+1}
\end{aligned}$$

$$\begin{aligned}
E(Y_{N(T)}^2) &= E(E(Y^2|N(T)=n)) \\
&= B^2\sum_{n=1}^{\infty} e^{-\lambda T}\frac{(\lambda T)^n}{n!} - 2(B-A)^2\sum_{n=1}^{\infty}\frac{1}{n+2}e^{-\lambda T}\frac{(\lambda T)^n}{n!} \\
&\quad -2A(B-A)\sum_{n=1}^{\infty}\frac{1}{n+1}e^{-\lambda T}\frac{(\lambda T)^n}{n!} \\
&= B^2(1-e^{-\lambda T}) - 2(B-A)^2\frac{1}{(\lambda T)^2}(\lambda T - 1 + e^{-\lambda T} - e^{-\lambda T}(\lambda T)^2/2) \\
&\quad -2A(B-A)\frac{1}{\lambda T}(1-e^{-\lambda T} - \lambda T e^{-\lambda T})
\end{aligned}$$

Using the formula $Var(Y_{N(T)}^2) = E(Y_{N(T)}^2) - E(Y_{N(T)})^2$ and collecting all terms, $Var(K(T)) = Var(Y_{N(T)}^2) = B^2 - A^2e^{-\lambda T} + \frac{(B-A)^2}{(\lambda T)^2} - (B - Ae^{-\lambda T} + (B-A)\frac{e^{-\lambda T}}{\lambda T})^2$. The work of collecting terms is tedious. Step-by-step details can be obtained from the author upon request. One way to double check the formula is to use the simulation algorithm listed in Section 4.

## Appendix C

The cumulative distribution functions and probability density functions of total benefit $K(T, \theta)$ (for the SRTM) and $K(N, \theta)$ (for the SRNB).

First I consider the distribution of $K(N, \theta)$. Using convolution, results in Table 1 and Eq. (1), the cumulative distribution function is

$$Pr(K(N,\theta) \leq k) = \int_0^\infty F_{\theta N}(k+ct)g_N(t)dt = \int_0^\infty \left(\frac{k+ct-A}{B-A}\right)^{\theta N} \lambda e^{-\lambda t}\frac{(\lambda t)^{N-1}}{(N-1)!}dt$$

Differentiating w.r.t. $k$, the probability density function is

$$p_N(k) = \int_0^\infty f_{\theta N}(k+ct)g_N(t)dt = \int_0^\infty \theta N \left(\frac{k+ct-A}{B-A}\right)^{\theta N-1} \frac{1}{B-A}\lambda e^{-\lambda t}\frac{(\lambda t)^{N-1}}{(N-1)!}dt$$

Then I consider the distribution of $K(T, \theta)$. The cumulative distribution function is

$$\begin{aligned} Pr(K(T,\theta) \leq k) &= \sum_{n=1}^{\infty} F_n(k+cT)Pr(N(T)=n) + I(k=-cT)Pr(N(T)=0) \\ &= I(A-cT \leq k \leq B-cT)\sum_{n=1}^{\infty}\left(\frac{k+cT-A}{B-A}\right)^{n} e^{-\theta\lambda T}\frac{(\theta\lambda T)^n}{n!} \\ &\quad +I(k=-cT)e^{-\theta\lambda T} \end{aligned}$$

$I(.)$ is the indicator function.

Differentiating w.r.t. $k$, the probability density function is

$$\begin{aligned} q_T(k) &= \sum_{n=1}^{\infty} f_n(k+cT)Pr(N(T)=n) + I(k=-cT)Pr(N(T)=0) \\ &= I(A-cT \leq k \leq B-cT)\sum_{n=1}^{\infty} n\left(\frac{k+cT-A}{B-A}\right)^{n-1} \frac{1}{B-A}e^{-\theta\lambda T}\frac{(\theta\lambda T)^n}{n!} \\ &\quad +I(k=-cT)e^{-\theta\lambda T} \end{aligned}$$

## Appendix D

Proof of Theorem 1.

(1) Let $X = \theta N = \theta\lambda T_{we}(N)$.

$$\begin{aligned} & AD(N,\theta) = E(K(N,\theta)) - E(K(T_{we}(N),\theta)) \\ = \;& B - \frac{B-A}{X+1} - B(1-e^{-X}) + \frac{B-A}{X}(1-e^{-X}-Xe^{-X}) \\ = \;& Ae^{-X} + \frac{B-A}{X(X+1)}(1-e^{-X}-Xe^{-X}) \\ > \;& 0 \end{aligned} \tag{29}$$

In the last step, the sum $\sum_{n=1}^{\infty} e^{-X}\frac{X^n}{n!} = 1-e^{-X}$ ,$e^{-X}$, $Xe^{-X}$ are the first two terms and all terms in the sum are positive, so $1-e^{-X}-Xe^{-X} > 0$.

(2)

$$\frac{\partial AD(N,\theta)}{\partial X} = -Ae^{-X} + (B-A)\frac{e^{-X}(X+1)^3-(2X+1)}{X^2(X+1)^2} \tag{30}$$

It's obvious that $-Ae^{-X} < 0$. When $X = 1$, $e^{-X}(X+1)^3-(2X+1) < 0$. Let $h(X) = e^{-X}(X+1)^3-(2X+1)$, $h'(X) = e^{-X}(X+1)^2(2-X)-2 < 0$ when $X \geq 1$. So $(B-A)\frac{e^{-X}(X+1)^3-(2X+1)}{X^2(X+1)^2} < 0$ and $\frac{\partial AD(N)}{\partial X} < 0$ when $X \geq 1$. $\frac{\partial AD(N)}{\partial \theta} = \frac{\partial AD(N)}{\partial X}\frac{\partial X}{\partial \theta} = \frac{\partial AD(N)}{\partial X}\lambda T_{we} < 0$. Due to the symmetry, $\frac{\partial AD(N)}{\partial \lambda} < 0$ and $\frac{\partial AD(N)}{\partial T_{we}} < 0$

(3) Using the results in Table 1, the mean and standard deviation of bid price distribution are $\mu_b = (B+A)/2$ and $\sigma_b = (B-A)/2\sqrt{3}$. Thus, $A = \mu_b - \sqrt{3}\sigma_b$ and $B-A = 2\sqrt{3}\sigma_b$. Plugging these results into Eq. (29),

$$\begin{aligned} AD(N,\theta) &= \mu_b e^{-X} + \left[\frac{2\sqrt{3}(1-e^{-X}-Xe^{-X})}{X(X+1)} - \sqrt{3}e^{-X}\right]\sigma_b \\ &= \mu_b e^{-X} + \frac{2-e^{-X}(X+2)(X+1)}{X(X+1)}\sqrt{3}\sigma_b \end{aligned} \tag{31}$$

When $X = 2$, $2-e^{-X}(X+2)(X+1) > 0$. Let $h_2(X) = 2-e^{-X}(X+2)(X+1)$, $h_2'(X) = e^{-X}(X^2+X-1) > 0$ when $X \geq 2$. So $2-e^{-X}(X+2)(X+1) > 0$ and $\frac{\partial AD(N,\theta)}{\partial \sigma_b} = \frac{2-e^{-X}(X+2)(X+1)}{X(X+1)}\sqrt{3} > 0$ when $X \geq 2$.

(4) From Eqs. (13) and (17) and Definition 4, $RR(N,\theta) = \frac{\theta N(B-A)^2}{(\theta N+1)^2(\theta N+2)} + \frac{c^2 N}{\lambda^2} - B^2 - A^2 e^{-\theta\lambda T} + \frac{(B-A)^2}{(\theta\lambda T)^2} - (B - Ae^{-\theta\lambda T} + (B-A)\frac{e^{-\theta\lambda T}}{\theta\lambda T})^2$.

$\frac{\partial RR(N,\theta)}{\partial c} = 2cN/\lambda^2 > 0$.

# Tables

| $F(X)$ | $f(X)$ | $F_n(Y)$ | $f_n(Y)$ |
|---|---|---|---|
| $d\frac{X-A}{B-A}$ | $\dfrac{1}{B-A}$ | $\left(\dfrac{Y-A}{B-A}\right)^n$ | $n\left(\dfrac{Y-A}{B-A}\right)^{n-1}\dfrac{1}{B-A}$ |
| $E(X)$ | $Var(X)$ | $E(Y_n)$ | $Var(Y_n)$ |
| $\dfrac{A+B}{2}$ | $\dfrac{(B-A)^2}{12}$ | $\dfrac{nB+A}{n+1}$ | $\dfrac{n(B-A)^2}{(n+1)^2(n+2)}$ |

Table 1: **Cumulative distribution functions, probability density functions, means and variances** of a uniform random variable $X \sim U(A,B)$ and the maximum value of $n$ i.i.d. uniform random variables $Y_n = max(X_1, ..., X_n)$.

| $\theta = 1$ | | | | | $\theta = 0.25$ | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $E(K)$ | $\sigma_K$ | $VaR_{0.99}$ | $ES_{0.99}$ | | $E(K)$ | $\sigma_K$ | $VaR_{0.99}$ | $ES_{0.99}$ |
| $N = 8$ | 94.83 | 2.62 | 86.37 | 84.99 | $N = 8$ | 89.27 | 5.95 | 74.92 | 73.91 |
| $T_{ce}(NA)$ | | | | | $T_{ce}(NA)$ | | | | |
| $N = 16$ | 93.72 | 1.84 | 88.26 | 87.10 | $N = 16$ | 90.20 | 4.25 | 77.88 | 76.23 |
| $T_{ce} = 1.55$ | - | 1.61 | 87.90 | 86.33 | $T_{ce}(NA)$ | | | | |
| $N = 32$ | 89.63 | 1.85 | 84.84 | 83.93 | $N = 32$ | 87.62 | 3.03 | 78.57 | 76.84 |
| $T_{ce} = 3.19$ | - | 0.78 | 86.83 | 86.07 | $T_{ce} = 3.00$ | - | 3.97 | 75.61 | 68.49 |
| $N = 64$ | 80.40 | 2.43 | 74.38 | 73.37 | $N = 64$ | 79.33 | 2.79 | 72.03 | 70.68 |
| $T_{ce} = 6.40$ | - | 0.39 | 79.03 | 78.64 | $T_{ce} = 6.27$ | - | 1.56 | 73.74 | 72.24 |

Table 2: **Means, standard deviations, value at risks and expected shortfalls** of stopping strategies $N = \{8, 16, 32, 64\}$ of the SRNB and their certainty equivalent stopping strategies of the SRTM . Calculations are based on 100,000 independent Monte Carlo simulation runs with parameter set $\{\lambda = 10, B = 100,000, A = 75,000, c = 3,000\}$. Confidence level $\alpha = 0.99$. Results of both perfect recall ($\theta = 1$) and partial recall ($\theta = 0.25$) are listed. Numbers are in thousands.

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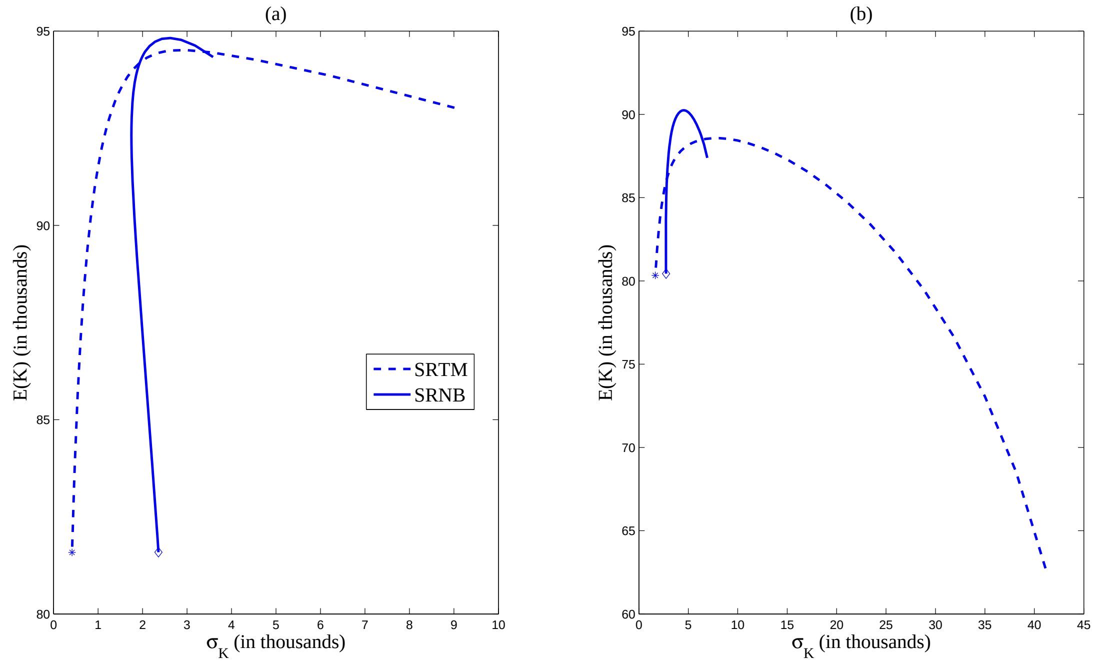


Figure 1: **Opportunity sets** of the SRTM and the SRNB. Parameter set of both panels is $\{\lambda = 10, B = 100,000, A = 75,000, c = 3,000\}$. Panel (a) shows the opportunity sets when recall is perfect ($\theta = 1$). Panel (a) shows the opportunity sets when recall is partial ($\theta = 0.25$).

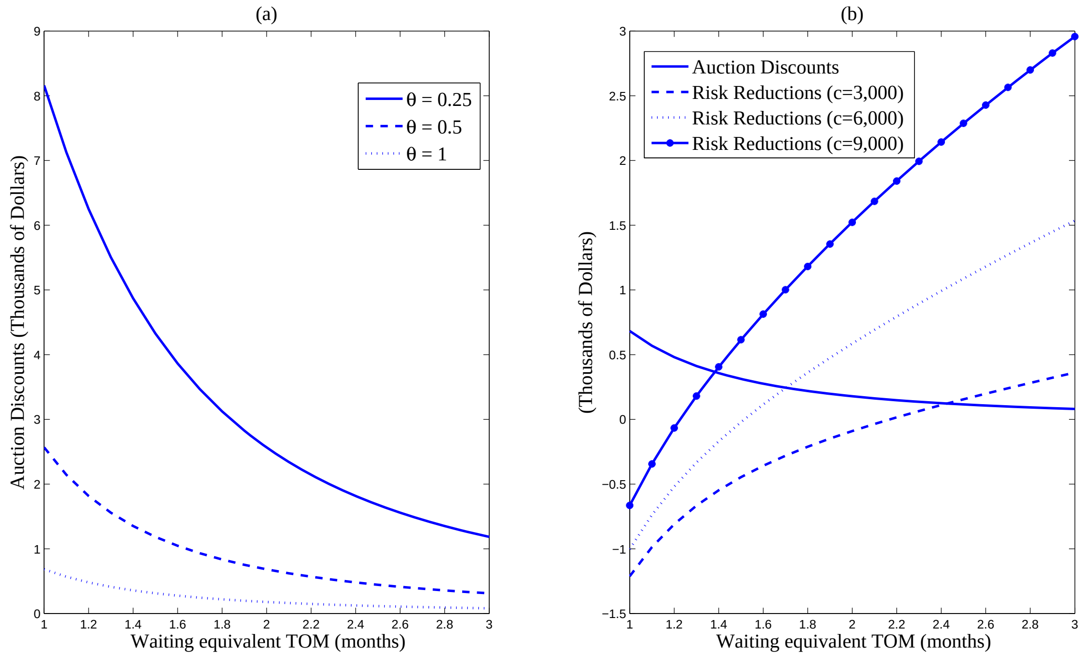


Figure 2: **Auction discounts and risk reductions.** Parameter set of both panels is $\{\lambda = 10, B = 100,000, A = 75,000\}$. For panel (a), $c = 3,000$; for panel (b), $\theta = 1$.

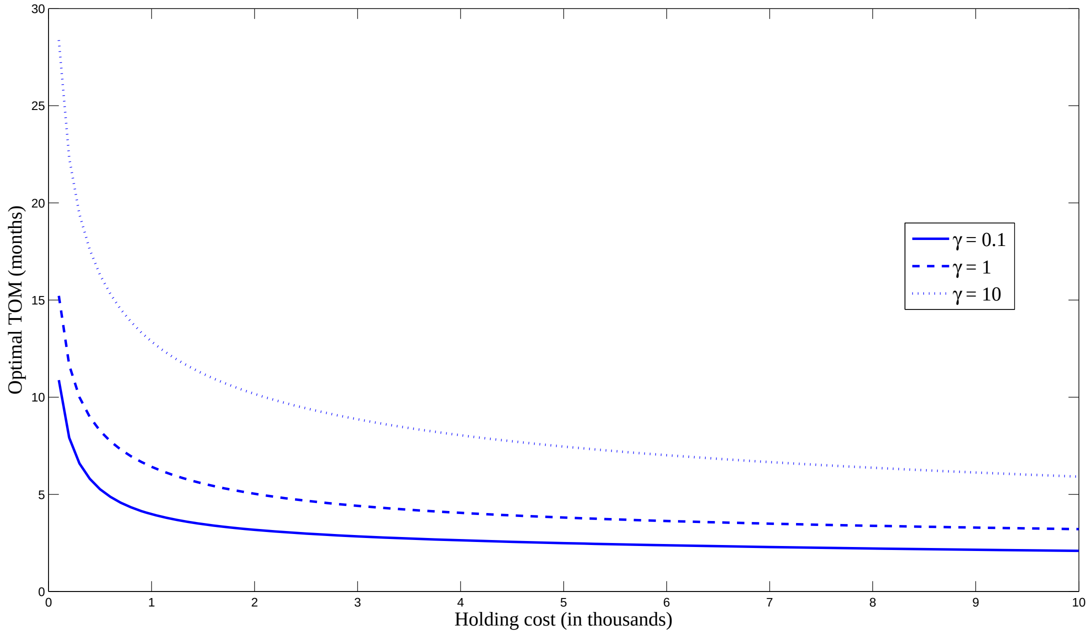


Figure 3: **Optimal time on market** when the holding cost and the seller's risk aversion changes. Parameter set is $\{\lambda = 10, B = 100,000, A = 75,000\}$.