Real Estate vs Stock Market: approaching the required rate of return through the Treynor and Black model

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Abstract

Real estate should be part of any well diversified portfolio. Its inclusion in an efficient portfolio is justified by its expected return and its risk features, among which it becomes remarkable their low correlation with financial assets, especially with common stock. Nevertheless, its low liquidity when compared with financial assets, its maintenance and transaction costs require that real estate enables investors to expect an alpha that absorbs these specific costs. On this basis we apply the Treynor and Black model to study their proportion in a portfolio constituted by real estate and common stocks. This model analyses the optimal combination between undervalued assets and the market index. The proportion of real estate in the new portfolio can be studied as the outcome of the combination between its own features and the properties of the market index. We perform a sensitivity analysis between the values of real estate weight and its alpha, which, after realizing that the specific risk of real estate is among the independent variables, gives way to a new relationship between the real estate proportion and its appraisal ratio. Next, we study the value created by the resulting new portfolio between real estate and the market index. The Sharpe ratio on the new portfolio is a function of the appraisal ratio of real estate, and, on this basis, we analytically relate the value created by the new portfolio to the appraisal ratio and the Sharpe ratio of the market index. Finally, we focus on the negative effect of the lack of divisibility of real estate on value creation, its consequences for bubbles and crisis, and on securitization as a way to overcome it.

1. Introduction

This work studies the properties of real estate to be combined with financial assets in an optimal portfolio. First of all we revise the variables that rule real estate investments. Next, we apply the Treynor and Black model to analyse the optimal percentage of real estate in a portfolio that combines it with the market index of the CAPM, and the patterns that determine value creation through this combination. A major feature of real estate investments is their indivisibility. For this reason, we study how it affects the optimal portfolio, and, particularly, the relationship between indivisibility and value. Last, we develop an analysis of bubbles based on exchange options.

2. Combining real estate with the market index of financial assets

Real estate is an asset outside the financial market. Its low correlation with financial assets, as stated in several works[1], constitutes an incentive to combine it with the market index of financial assets in order to create value. As known, the lower the correlation, the higher the diversification. Hudson-Wilson(2002) point out five advantages that stem from combining real estate and financial assets in the investor's portfolio: risk reduction, higher return, hedging against unexpected inflation, the portfolio reflects best the available assets, and higher cash flows than the ones provided by stocks and bonds. Nevertheless, real estate also has its drawbacks as an investment asset. Its liquidity is lower than the liquidity that financial assets have in secondary markets, and, in addition, its transaction costs are substantially higher. Thus, the inclusion of real estate in a portfolio only makes sense if it provides an alpha that compensates these drawbacks.

We will approach the mixing between financial and real estate assets by means of the Treynor and Black (1973) model, The Treynor and Black model tackles the problem of building up a portfolio that combines a portfolio of undervalued assets with the market index. We substitute the portfolio of undervalued assets by real estate, and, on this basis, we explore the properties of such a combination. This change leads to a central difference in the analysis: the short run that stems from dealing with undervalued assets gives way to the long run that stems from considering a stable portfolio that enlarges the market index of financial assets by adding real estate to it.

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[1] See for example Brueggeman and Fisher (2007) p. 602

2.1 The optimum percentage of real estate in the portfolio of real estate and financial market index

2.1.1 The optimum percentage of real estate as a function of alpha

Be *M* the market index of financial assets, *p* the real estate portfolio and *m* the enlarged portfolio that combines both. The percentage of real estate in the enlarged portfolio is denoted by *w*. The expected rate of return and the corresponding standard deviation for the enlarged portfolio are:

$$\bar{R}_m = w\bar{R}_p + (1-w)\,\bar{R}_M \tag{1}$$

$$\sigma_m = \left[w^2\sigma_p^2 + (1-w)^2\,\sigma_M^2 + 2w(1-w)\sigma_{pM} \right]^{1/2} \tag{2}$$

And the Sharpe ratios of the market index and the enlarged portfolio:

$$S_M = \frac{\bar{R}_M - r}{\sigma_M} \tag{3}$$

$$S_m = \frac{\bar{R}_m - r}{\sigma_m} \tag{4}$$

The value of *w* that maximizes the Sharpe ratio of the enlarged portfolio is[2]:

$$w^* = \frac{\left[\bar{R}_p + r(\beta_p - 1) - \bar{R}_M\beta_p\right]\sigma_M^2}{[\beta_p(2r - \bar{R}_M - \bar{R}_p) + \bar{R}_p - r]\sigma_M^2 + (\bar{R}_M - r)\sigma_p^2} \tag{5}$$

Performing some algebraic operations on this equation, we arrive at:

$$w^* = \frac{\alpha_p\sigma_M^2}{\sigma_{\varepsilon p}^2\left(\bar{R}_M - r\right) + \alpha_p\left(1-\beta_p\right)\sigma_M^2} \tag{6}$$

Where, α_p means the expected excess return of real estate with respect to the required rate of return that stems from the financial market:

$$\alpha_p = \left(R_p - r\right) - \left(R_M - r\right)\cdot\beta_p \tag{7}$$

Equation (6) shows that the optimal percentage of real estate in the enlarged portfolio depends on:

a) The expected market index risk premium

[2] This result is obtained by substituting (1) and (2) into (4), deriving with respect to *w*, equating the derivative to zero and clearing *w*. Bodie, Kane and Marcus (2002 p.926 eq. 8.7) apply this equation to the analysis of active strategies.

b) The total risk of the market index

c) The beta of real estate

d) The alpha of real estate

e) The specific risk of real estate

Studying the sensitivity of w^* with respect to these variables, we obtain that de signs of its derivatives are as follows[3]:

	$\overline{R}_M - r$	σ_M^2	β_p	α_p	$\sigma_{\varepsilon p}^2$
w^*	negative	positive	positive	positive	negative

From (6) we obtain the value of alpha that fits with a determinate percentage of real estate in the enlarged portfolio, i.e. it enables to calculate the required alpha for a specific percent of real estate in the enlarged portfolio. Let us call $\alpha_p *$ to this alpha value:

$$\alpha_p^* = \frac{w^* \sigma_{\varepsilon p}^2 (R_M - r)}{\sigma_M^2 \left[1 - w^*(1 - \beta_p)\right]} \tag{8}$$

Which can be written as:

$$\alpha_p^* = S_M \frac{\sigma_{\varepsilon p}}{\sigma_M} \frac{1}{\left[1 - w^*(1 - \beta_p)\right]} \sigma_{\varepsilon p} \tag{9}$$

This result shows that a higher optimum percentage in real estate requires a higher alpha. The required alpha also increases with the Sharpe ratio of the market index, the ratio between the specific risk of real estate and the volatility of the market index and the specific risk of real estate.

[3] Derivatives are calculated in the appendix.

2.1.2 The appraisal ratio

In (6) we have alpha and the specific risk ($\sigma_{\varepsilon p}$) of real estate as independent variables. Let us recall that their quotient is the appraisal ratio (*AR*):

$$AR_p = \frac{\alpha_p}{\sigma_{\varepsilon p}} \tag{10}$$

Writing the optimal percentage of real estate as a function of the appraisal ratio, we obtain:

$$w^* = \frac{AR_p}{\left(\frac{\sigma_p^2}{\sigma_M^2} - \beta_p{}^2\right)^{\frac{1}{2}} \frac{\left(\overline{R}_M - r\right)}{\sigma_M} + AR_p\left(1-\beta_p\right)} \tag{11}$$

Introducing the Sharpe ratio of the financial market index and recalling the relationship between total risk, systematic and specific risk, we have:

$$w^* = \frac{AR_p}{\frac{\sigma_{\varepsilon p}}{\sigma_M} S_M + AR_p\left(1-\beta_p\right)} \tag{12}$$

Thus, the optimal percentage of real estate in the enlarged portfolio is a function of:

a) A variable that only depends on real estate: its appraisal ratio.
b) A variable that only depends on financial assets: the Sharpe ratio of its market index.
c) Two variables that depend on real estate and financial assets at the same time: the coefficient beta of real estate and the ratio between the specific risk of real estate and the risk of the market index. .

Let us call μ this standard deviations ratio that can be interpreted as a penalty for lack of diversification. Then we have

$$w^* = \frac{AR_p}{\mu S_M + AR_p\left(1-\beta_p\right)} \tag{13}$$

Studying the sensitivity of $w*$ with respect to the mentioned variables, we obtain that the signs of its derivatives are as follows:

	AR_p	S_M	μ	β_p
w^*	positive	negative	negative	positive

2.2 Nil correlation coefficient

Empirical studies have shown that the correlation coefficient between real estate and financial assets is usually very low. Let us explore the consequences of a nil correlation coefficient, which makes beta equal to zero. In this case, the optimum percentage becomes:

$$w^* = \frac{AR_p}{\eta S_M + AR_p} \tag{14}$$

And taking into account that in this case the appraisal ratio turns out to be equal to the real estate Sharpe ratio, real estate alpha equates its own risk premium and its specific risk is at the same time its total risk, we have:

$$AR_p = S_p \tag{15}$$

and,

$$w^* = \frac{S_p}{\eta S_M + S_p} \tag{16}$$

In this specific case, the optimum percentage of real estate shows a positive sensitivity with the Sharpe ratio of real estate, and negative sensitivities with respect to the Sharpe ratio of the financial market and the standard deviations ratio (μ).

3. The Sharpe ratio of the enlarged portfolio and risk premium

Bodie, Kane and Marcus (2002, p. 816) show that the Sharpe ratio of the enlarged portfolio that stems from the Treynor and Black model turns out to be:

$$S_m^2 = S_M^2 + AR_p^2 \tag{17}$$

i.e.:

$$S_m = \sqrt{S_M^2 + AR_p^2}$$

(18)

After some algebraic operations, the increase that it has had due to the incorporation of real estate can be written as:

$$S_m - S_M = S_M \left(\sqrt{1 + \frac{AR_p^2}{S_M^2}} - 1 \right) \tag{19}$$

which is the increase in the risk premium per unit of standard deviation.

The Capital Market Line enables to calculate the increase in investor's risk premium created by the enlarged portfolio. It evolves from:

$$\bar{R} = r + S_M \sigma \tag{20}$$

To:

$$\bar{R} = r + S_m \sigma = r + \sqrt{S_M^2 + AR_p^2}\,\sigma \tag{21}$$

Thus the new risk premium is:

$$\bar{R} - r = \sqrt{S_M^2 + AR_p^2}\,\sigma \tag{22}$$

And its increase has been:

$$\Delta\left(R - r\right) = S_M \left(\sqrt{1 + \frac{AR_p^2}{S_M^2}} - 1 \right) \cdot \sigma \tag{23}$$

After this result we can say that the increase in the risk premium depends on a sole variable connected with real estate: its appraisal ratio.

4. Value creation

Let us revise the value creation that stems from investing in real estate under the hypothesis of this work. We assume that the required rate of return depends on the market index of financial assets, and, thus, the required rate of return of well diversified portfolios depends on the security market line. The value of such a portfolio at one year horizon would be:

$$V = \frac{1 + r + \dfrac{\bar{R}_m - r}{\sigma_m}\sigma}{1 + r + \dfrac{\bar{R}_M - r}{\sigma_M}\sigma} \tag{24}$$

i.e.:

$$V = \frac{1 + r + S_m \sigma}{1 + r + S_M \sigma} \tag{25}$$

The increase in value with respect of having combined the financial market index with the real estate asset has been:

$$\Delta V = \frac{1+r+S_m\sigma}{1+r+S_M\sigma} - 1 = \frac{(S_m - S_M)\sigma}{1+r+S_M\sigma} \tag{26}$$

$$\Delta V = \frac{S_M\sigma\left(\sqrt{1+\frac{AR_p^2}{S_M^2}}-1\right)}{1+r+S_M\sigma} \tag{27}$$

$P_{\min}$ Thus, the increase in value depends on the same variables as the increase in the risk premium. The contribution from real estate to the increase in value is encapsulated on its appraisal ratio.

5. Indivisibility vs. securitization

5.1 Indivisibility

Real estate lacks the divisibility of financial assets. Investing in real estate requires a minimum budget, be it $P_{\min}$. We assume at the same time that investing in real estate a higher amount than $P_{\min}$ is a continuous function, i.e. the indivisibility holds for investments lower than $P_{\min}$, but does not hold for higher amounts. Next, we explore the consequences of this property for the combinations between the enlarged portfolio and the risk free security.

Taking $P_{\min}$ into account, the minimum budget an investor must have in order to distribute his investment in the proportions *w* in real estate and (1-*w*) in the market index is:

$$B_{\min} = \frac{P_{\min}}{w} \tag{28}$$

Be an investor, that we denote as investor i, who wishes a volatility lower than the volatility of the enlarged portfolio:

$$\sigma_i < \sigma_m \tag{29}$$

Let us denote by x_i the proportion between both volatilities:

$$\sigma_i = x_i\sigma_m \tag{30}$$

Where:

$$x_i < 1 \tag{31}$$

The budget the investor must have in order to make the volatility of his portfolio equal to σ_i is:

$$B_i = \frac{B_{\min}}{x_i} \tag{32}$$

Because the distribution of this budget between the risk free security (lending) and the investment in the enlarged portfolio makes the latter equal to its minimum value:

a) Investment in risk free security: $(1 - x_i) B_i$

b) Investment in the enlarged portfolio:

$$x_i B_i = B_{\min} \tag{33}$$

Assuming, as stated, that any amount higher than $P_{\min}$ can be invested in real estate, this restriction does not hold for investors who aim to assume a volatility higher than σ_m.

Thus, the minimum budget enables investors to assume σ_m and higher volatilities by investing in the enlarged portfolio and combining it with borrowing at the risk free interest rate. Nevertheless, the minimum budget that investors who aim to assume a volatility σ_i lower than σ_m must have is:

$$B_i = B_{\min} \frac{\sigma_m}{\sigma_i} \tag{34}$$

as it stems from substituting x_i in (33).

Investors who do not have the budget to access the combinations between the enlarged portfolio and the risk free security, must remain on the traditional Capital Market Line that combines the stock market index with lending and borrowing. These investors lose the expected differential rate of return between both lines that can be expressed as:

$$\overline{R}_{im} - \overline{R}_{iM} = \left[\frac{\overline{R}_m - r}{\sigma_m} - \frac{\overline{R}_M - r}{\sigma_M} \right] \sigma_i = \left[S_m - S_M \right] \sigma_i \tag{35}$$

Thus, the loss of expected return consists of the difference between the Sharpe ratios multiplied by the standard deviation that mirrors the investor's risk preferences. In (35), $\overline{R}_{im}$ denotes the expected rate of return that stems from combining the enlarged portfolio with the risk free interest rate, and $\overline{R}_{iM}$ denotes the rate of return than the Capital Market Line enables to expect for the same volatility.

5.2 Securitization

The value that real estate indivisibility deprives investors to obtain can be recouped through securitization. It consists of substituting the real estate assets of the enlarged portfolio by securities of real estate investments, like REITs. Nevertheless, often REITs show a higher correlation with the market index than real estate. This increase in correlation is often attributed to the speculation fostered by the lower transaction costs of REIT securities with respect to real estate. Such increase in the correlation coefficient lessens the value created through the incorporation of real estate in the portfolio.

Next, we find the critical value of the correlation coefficient for which the value destroyed equates the value created.

Be β_p critical the value of β_p that equates α to zero:

$\beta_{critical} \Rightarrow \alpha = 0$

Taking into account the value of alpha, we can write:,

$$\left[\overline{R}_p - r\right] - \left[\left(R_M - r\right)\beta_p\right] = 0 \tag{36}$$

Hence the value of beta that equates alpha to zero is,

$$\beta^* = \frac{\overline{R}_p - r}{R_M - r} \tag{37}$$

And its corresponding correlation coefficient,

$$\rho^* = \frac{\overline{R_p} - r}{R_M - r}\frac{\sigma_M}{\sigma_p} \tag{38}$$

Thus, the necessary condition for creating value through securitization is that, after its introduction, the correlation coefficient between the real estate securities and the stock market index does not be higher than the value obtained in (38).

6. Bubbles

Akerlof and Shiller (2009) in his recent book *Animal Spirits*, argue that the current financial crisis was driven by speculative bubbles in the housing market, among other markets. In this section we analyze the role of indivisibility, one of the central features of real estate, in bubbles and crisis, and, next, we propose an indicator to detect bubbles.

6.1 Indivisibility, bubbles and crisis

The indivisibility of real estate investments makes bubbles bigger and crisis deeper. A simple model shows this assertion. Let us recall that the minimum budget an investor must have in order to distribute his or her investment in the proportions *w* in real estate and (*1*-*w*) in financial assets is:

$$B_{\min} = \frac{P_{\min}}{w}$$

The sensitivity of the proportion of real estate to the appraisal ratio is crucial now. Let us underline that the sign of its derivative, $(\frac{\partial w^*}{\partial AR})$, is positive, as stated in 2.1 and shown in the appendix. This means, at the same time, that the derivative of B_{min} with respect to *AR* is negative.

In a bubble the appraisal ratio, *AR*, obviously increases (otherwise we are not in a bubble). Thus the proportion of capital invested in real estate, *w*, increases as well, and, as it stems from(39), the minimum total budget the investor must have in order to include real estate in his portfolio, $B_{min,}$ decreases. The reduction of B_{min} makes real estate investment available to more investors, which stimulates the demand of real estate, makes the prices higher, and, therefore, the appraisal ratio higher as well. This increase in prices is strictly due to demand pressure, not to an increase of value, which is one of the central features of a bubble. To sum up:

$$\boxed{\Delta AR \Rightarrow \Delta w \Rightarrow \nabla B_{\min} \Rightarrow \Delta \text{ demand of real estate} \Rightarrow \Delta AR}$$

The reverse situation takes place in a crisis. *AR* decreases, *w* increases and B_{min} increases. The investors in real estate with lower budgets are stimulated to abandon real estate concentrating their investments in financial assets again. The offer of real estate increases due to this fact and stimulates prices going down. To sum up:

$$\boxed{\nabla AR \Rightarrow \nabla w \Rightarrow \Delta B_{\min} \Rightarrow \nabla \text{ demand of real estate} \Rightarrow \nabla AR}$$

As stated, securitization of real estate through REITs and specialized funds can be regarded as a way of reducing the indivisibility effect.

6.2 Detecting bubbles

A bubble consists of an overvaluation of a kind of assets. We aim to find an indicator to detect the overvaluation of a kind of assets with respect to the market index. A necessary condition for an asset to be overvalued at the end of a period is that it has obtained a positive anomalous return (i.e. positive Jensen alpha) not justified by facts or a change of expectations. If this alpha is statistically significant, the condition is reinforced. Obviously, the challenge is to evaluate the weight of the change of expectations.

Next, we propose an indicator of overvaluation through exchange options. Be a strategy of exchanging the final value of 1 € invested in the market index for the final value of 1 € invested in real estate (i.e. to deliver the market index and receive real estate) without risk of losing in the exchange and without investing. This strategy consists of an option to exchange the market index for real estate, which is valued according to Margrabe (1978) as shown in the appendix. The option makes the exchange riskless. In order to avoid investing, the initial price of the option is to be financed through a credit at the risk free interest rate. The cash-flows of this strategy are synthesised in the next table:

	t=0	t=1
Purchase of exchange options	$-CH_0$	$Max\ \{0, R_A-R_M)$
Credit	CH_0	$-\ CH_0\ (1+r)$
	0	$Max\ \{0, R_A-R_M)\ -\ CH_0\ (1+r)$

In this strategy, the risk premium is embedded in the initial value of the exchange option. Thus, at then end of the period an extra return has been obtained if the difference between the rates of return of *A* and *M* turns out to be higher than the initial value of the exchange option (initial values of underlying assets equal to 1€) capitalized at the risk free interest rate. As an extra return, it can indicate a bubble to the extent that it is not due to a change of expectations or unexpected facts. Denoting by *b* the bubble indicator, we can write:

$$b = (R_A - R_M) - CH_0 \cdot (1 + r) \qquad (40)$$

This analysis has the advantages of being based on the reliability of option valuation, for instance it is independent from risk attitudes, and enabling a direct comparison between the two rates of return under consideration.

Conclusions

The expected rate of return of real estate has to add to the rewards for time and risk an additional reward to compensate for the lack of liquidity and the higher transactions amount of real estate with respect to financial assets. This extra return can be regarded as a compensatory alpha that equilibrates real estate with financial assets. If real estate offers a genuine alpha beyond this compensatory alpha, it can be advantageously combined with the market index. The features of the enlarged portfolio that stems from combining real estate with the market index can be analyzed by means of the Treynor and Black model.

The weight of real estate in the enlarged portfolio depends on its appraisal ratio, its beta and its specific risk, apart from the variables of the market index. Nevertheless, the real estate variables that determine the parameters of the enlarged portfolio are reduced to the appraisal ratio when we focus on risk premium and to the increase in value obtained by building up the enlarged portfolio. Thus, the appraisal ratio can be regarded as the driver through which real estate creates value.

The indivisibility of real estate investments requires an increasing budget in order to combine the enlarged portfolio with lending, i.e. investing in risk free assets. In this case, the lower the volatility, the higher the budget the investor needs to maintain the optimal positions in real estate, the financial market index and the risk free assets. Securitization constitutes a way of overcoming the effects of indivisibility.

The indivisibility of real estate or, in other words, the fact that to invest in real estate requires a minimum budget contributes to boost bubbles and to make crisis deeper. In bubbles, it provokes an increase in the optimum percentage invested in real estate, and in crisis it generates the reverse effect.

Finally, we have proposed an indicator of overvaluation through exchange options that can be applied to detect bubbles.

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